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TRISEQUENTIAL PROPOSITIONAL CALCULUS

Introduction

The method of sequents for the classical logical calculi has been made the first time by G. Gentzen [5] for formalizing the concept of proof which occurs in deductional practice. In some way, dual method to Gentzen's sequential method is so called tableau method or method of dyadic trees. This method was invented independetly by: E. Beth [2], J. Hintikka [7] and K. Schütte [17].

In sequent method of constructions of logical calculi the main conception is sequent. Sequent is defined as a pair or finite sequence of finite sets of formulae. In method of dyadical trees essence is employing of marking of formulae and prooving of tautological formulae (tautology of formulae), it is based on the overfilling of formulae „marked noncorrect".

The method of sequents for construction of n-value logics define V. G. Kirin [8], G. Rousseau [12], [13], [14] and Z. Saloni [15], [16].

Here the sequent is defined and interpreted in way which is similar to the way in 2 – valued logic. In logical calculi which occured above is needed some numbers of logical one – argument and constans.

The method of trees or tableau method – n – value propositional calculi was introducted by S. J. Surma in works [19] and [20]. This method like in case of two – valued logic needs the conception of marked formula, or some finite set of generators from metalanguage which is the same. The purpose of this work is to show for the trisequential logics so – called trisequents method dually to the method of Surmaš trees.

This method does not demand to employ the operators of marking the formulae. And also we do not demand constans and 1 – argument concectives . The main conception of this method is trisequent, which is equivalent to sequent for instance from works [10], [12], [15] with another interpretation.

1. Algebraic preliminaries

Let $\Omega=(\Omega_m: m\in N)$ be a familly of sets of operation symbols (cf. Cohn [5]).

By an Ω – algebra we mean an ordered pair $A=(A, op)$, where A is a set and $op=(op_m: \Omega_m \to A^{(A^m)}: m\in N)$ is a family of functions for $A^{(A^m)}$ denoting the set of functions $f: A^m \to A$.

Let V be a denumerable set of propositional variables. We define the valuation of Ω – terms over V in the Ω – algebra $A=(A,(op_m: m\in N))$ to be a unique function satisfying the following conditions:

$$[x]\alpha = v(x) \text{ for every } v\in (A)^v$$

$$[w(t_1,t_2,\ldots\ldots,t_m)](v)=op_m(w)([t_1](v),[t_2](v),\ldots,[t_m](v))$$

for all $w\in\Omega_m, m\in N, v\in(A)^v$

2. Propositional S – calculus (semantical part)

Let $\underline{3}=\{1,2,3\}$. By a specification we mean an ordered triple

$$S=(\underline{3},D,C),$$

where $D\subseteq 3$ is a set of distinguished elements and

$$C\subseteq \bigcup_{m\in N} \underline{3}^{(\underline{3})^m} \qquad \text{for C being a set.}$$

By a type determined by specification $S=(\underline{3},D,C)$ we mean a family $\Omega^S = (\Omega^S_m: m\in N)$ of disjoint sets such that

$$\Omega^S_m = C\cap \underline{3}^{(3)^m} \text{ for every } m\in N$$

By a metrical algebra associated to a specification $S=(\underline{3},D,C)$ we mean an Ω^S – algebra.

$$A_S=(\underline{3},(op_m: m N)) \text{ such that } op_m(f)=f$$
$$\text{for every } f\in\Omega^S_m, m\in N$$

By a propositional calculus (more precisaly propositional S – calculus) we mean an ordered quadruple

$P_c=(S,L,[?],Pr)$ where

(i) $S=(\underline{3},D,C)$ is a specification,

(ii) L, called a language of P_c, is an ordered pair $L=(V,F)$ such that V is a denumerable set of variables, F is the set of Ω – terms called the set of formulae.

(iii) $[?]: \underline{3}^V \times \Omega^S$ – term $\to \underline{3}$

$[(v,x)]=v(x)$

$[(v,F(t_1,t_2,\ldots,t_m))]=f([(v,t_1)], [(v,t_2)],\ldots,[(v,t_m)])$

(iv) data contained in Pr will be defined in the next subsetion. Pr contains an information about mechanism for acceptation or refutation of formulae.

Let $P_c=(S,L,[?],Pr)$ be propositional S – calculus, where $S=(\underline{3},D,C)$ and $L=(V,F_S)$. Let $i \in D$. By semantical i – tautology of propositional S – calculus we mean a formula $a \in F_S$ such that for every $v \in \underline{3}^V$

$$[a](v)=i$$

3. Propositional calculus (formal part)

Let X_1, X_2, X_3 be finite sets, some of them may be empty of formulae, i, e.

$$X_i \subseteq F_S \text{ for } i=1,2,3.$$

A sequent is an ordered 3 – triple (X_1, X_2, X_3) which will be denoted $X_1+X_2+X_3$. We will write X_i, a for $X_i \cup \{a\}$. The sequents will be denoted by Σ with indices if necessary.

By an overfilled sequent we man a sequent $\Sigma=X_1+X_2+X_3$ such that $X_j \cap X_k \neq 0$ some j, k in$\{1,2,3\}$.

To present the rules given in the sequel in a concise form we shall adopt the following notation.

Let $\Sigma=X_1+X_2+X_3$ be a sequent and $a_1, a_2, \ldots, a_m$ be Ω – term, $x \in \underline{3}_m$ and let $F \in \Omega^S_m$. Let moreover $\Sigma_x=X'_1+X'_2+X'_3$ be a sequent such that

$$X'_i=X_i \cup \{a_k : p^r_k(x)=i\}$$

Then the scheme of the introduction rule for the connective F to $X_j=1,2,3$ of the sequent $\Sigma=X_1+X_2+X_3$ will be the following:

$$(r)\frac{\{\Sigma_x : x \in \underline{3}^m, (op^S_m(F))(x)=j\}}{\ldots\ldots+X_j, F(a_1, a_2, \ldots, a_m)+\ldots}$$

for $j=1,2,3$.

By an unordered tree we shall mean a collection $T=(D,D',1,R,x_1)$ such that:

(1) A set D, of elements called points, the set $D' \subset D$:

(2) A function, 1, which assings to each point $x \in D$, s positive integer $l(x)$ called the level of x,

(3) A relation $R \subset D^2$: xRy we read „x is a predecessor of y" or „y is successor of x",

This relation must obey the following conditions:

(i) There is a unique point x_1 of level 1. This point we call the origin of the tree.

(ii) Every point $x \in D$, $x \neq x_1$ has at most a unique successor

(iii) For any point x, y, if y is a successor of x then $l(x)=l(y)+1$

(iv) $D'=\{y : \{x : xRy\}=0\}$

(v) $\text{card}\{x : xRy\} \leqslant 3^m$

By a proof tree in the sequential calculus we mean a tree $T=(P,P',1,R,x_1)$ where:

(1) P is the set of sequents
(2) $P' \subset P$ and P' is the set overfilled sequents
(3) The relation R is defined by following equivalence $\Sigma_1 R \Sigma_2$ there exists the rule $r \in (r)$ such that Σ_2 is the conclusion of the rule r and Σ_1 is one of its premises

The sequent $\Sigma = X_1 + X_2 + X_3$ is a terminal sequent if there exists a formula $a \in \Omega^S \rightarrow$ term and $j, 1 \leq j \leq 3$ so that for every $i, 1 \leq i \leq 3, i \neq j, X_i = 0$ and $X_j = \{a\}$. A propositional formula a is a theorem in the trisequential logic if and only if there exist sets of the overfilled sequents from which there are proofs of the following terminal sequents.

$$\Sigma_i = X_{i1} + X_{i2} + X_{i3}$$

$$\text{for every } i,\ 1 \leq i \leq k,\ 2 \leq k \leq 3$$

Theorem: The formula a is a theorem of the trisequential propositional calculus if and only if a is a tautology.

Proof.

Let first a be unprovable, and assume it is so because in each proof – tree with terminal sequent

$$a+0+0$$

there is a non – overfilled initial sequent – the other case can be handled similarly. Now let T be a maximal proof – tree with a+0+0 as a terminal sequent. The maximality implies that in T the initial sequents do not contain logical connectives and the assumption implies that there exists a maximal branch B of T consisting of non – overfilled sequents only. Let the sequents of B be in descending order

$$\Sigma^1, \Sigma^2, \Sigma^3, \ldots, \Sigma^m = a+0+0$$

Then for each $1 \leq k \leq m, \Sigma^k$ appears as one of the premises in some scheme rule the conclusion of which is Σ^{k+1}

Now let us define the valuation $v: V \rightarrow \underline{3}$ as follows.

For proposition variables not appearing in $\Sigma = X_1^1 + X_2^1 + X_3^1$ we fix their values arbitrarily, otherwise $v(p) = i$ if and only if $p \in X_i^1$. Since Σ^1 is not overfilled , this defines univocally. Moreover Σ^1 contains no logical connectives, therefore the following claim holds for $k=1$: Claim. Let $\Sigma^k = X_1^k + X_2^k + X_3^k$.

For each $i \in \underline{3}$, if $b \in X_i^k$ then $[b](v) = i$.

Next we check the Claim for each $k \leq m$. Suppose it holds for some $k < m$. Since Σ^{k+1} is the conclusion of a schema rule and Σ^k is of its premises, $b \in X_i^{k+1}$ implies $b \in X_i^k$ except for exactly one formula b. Let's assume $b \in X_i^{k+1}$. Now if, $b = w(a_1, a_2, \ldots, a_r)$ then by the definition of the schema rules $a_1 \in X_{j_1}^k$, $a_2 \in X_{j_2}^k, \ldots, a_r \in X_{j_r}^k$ for some $j_1, j_2, \ldots, j_r \in \underline{3}$ such that $op_r^s(w)(j_1, j_2, \ldots, j_r) = i$.

Now the induction hypothesis gives $[a_1](v) = j_1, [a_2](v) = j_2, \ldots, [a_r](v) = j_r$ thus

$$[b](v)=(op^s_r(w))([a_1](v),[a_2]\ (v),\ldots,[a_r]v))$$

proving the Claim for k+1.

Since $\{a\}=X^m_1$, for k=m the Claim gives [a](v)=1 which shows that a is not s tautology. Therefore if the formula a is a tautology then it is a theorem of the trisequential propositional calculus.

To prove the opposite implication, suppose that a is provable, but it is not tautology. Then there exists a valution $v: V \to \underline{3}$ such that $[a](v)<1$ for $2 \leq 1 \leq 3$. It can be checked similarly as above, that if the Claim holds for some sequent in the derivation – tree, then it holds for at least one of its predecessor sequents. Consequently, there is, a maximal branch in the proof – tree on which the Claim holds. Then this branch cannot contain an overfilled sequent, which is a contradiction.

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STRESZCZENIE

Celem tej pracy jest pokazanie trójsekwentowej metody dla trójwartościowego rachunku zdań, która jest dualną metodą do tzw. „drzew Surmy". Metoda ta nie wymaga dodatkowych operatorów oznaczających formuły. Podstawowym pojęciem w tej metodzie jest trójsekwent, równoważny temu pojęciu używanemu przez innych autorów, jednakże ma inną interpretację.