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The summarized volume distribution in $GI/M/1/\infty$ queueing system

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The problem of summarized demands volume characteristics determination for queueing systems and application of the theory of queues with random demands length for memory space defining was discussed in many papers [1-3]. But in previous publications (except [3]) queues with Markov arrival flow were analyzed. In [3] paper a stationary queue of general kind without losses was considered. For this queue only the first moment of summarized demands volume was determined.

The present paper deals with $GI/M/1/\infty$ queue. Denote as $A(t)$ the distribution function of the time interval between neighboring moments of demands entering to the system, let $\alpha(q)$ be the Laplace-Stieltjes transform of this function. Let the first moment

$$\alpha_1 = -\alpha'(0) = \int_0^\infty t dA(t)$$

exists and $\lambda = 1/\alpha_1$. Let μ be a parameter of a service time ($\mu > 0$) and $\rho = \lambda/\mu < 1$ condition takes place, i.e. for the queue under discussion stationary mode exists. As it is known [1], generally for such queues the service time depends on the demand's length. As far as the service time has an exponential distribution, this fact leads to substantial restrictions of the character of this dependency. It is naturally to consider the next cases: 1) demand's length has an arbitrary distribution with distribution function of $L(x)$ and demand's service time not depends on the demand's length, 2) demand's length ζ has an exponential distribution with $f > 0$ parameter, it's service time ξ is proportional to the length ($\xi = c\zeta, c > 0$) and so has an exponential distribution too with a parameter of $\mu = f/c$. In the first case the determination of demands summarized volume characteristics is very simple. It is easy to determine, that the Laplace-Stieltjes transform $\delta(s)$ of stationary summarized volume (if stationary mode exists) in this case is

$$\delta(s) = 1 - \frac{\rho(1-\varphi(s))}{1-\kappa\varphi(s)},$$

where $\varphi(s)$ is the Laplace–Stieltjes transform of $L(x), \kappa$ is a single real solution of $\kappa = \alpha(\mu - \mu\kappa)$ equation, such as $0 < \kappa < 1$ [4].

Below we consider the second case in detail. Denote as σ the stationary summarized volume (the total sum of demands lengths in the system). Denote the distribution function of σ random variable as $D(x)$, let $\delta(s)$ be the Laplace-Stieltjes transform of $D(x), \chi$ be the total sum of service times of demands present in the system in stationary mode. In the case under discussing we have $\chi = c\sigma$. Let $R(t) = D(t/c)$ be the distribution function of χ random variable and $r(q) = \delta(cq)$ be the Laplace-Stieltjes transform of $R(t)$ function.

The main purpose of the paper is the determination of $\delta(s)$ function, or (it is the same) $r(q)$ function.

Let η be the number of demands present in the system in stationary mode. It is known [4], that $p_0 = \mathbf{P}\{\eta = 0\} = 1 - \rho, p_k = \rho(1-\kappa)\kappa^{k-1}, k = 1, 2, \ldots$. Let v be the number of demands, which are in the system at the time moment just before the moment of demand's entering to the system in stationary mode. It is known [4], that $\pi_k = \mathbf{P}\{v = k\} = (1-\kappa)\kappa^k, k = 0, 1, \ldots$. Let ξ^* be the time interval between beginning of service for a demand which is on service at time moment τ and the moment of τ. It is obviously, that the total sum χ of service time of demands being on service at the time moment τ may be represented as $\chi = \chi_1 + \chi_2$, where χ_1 is a total sum of service time of demands waiting for the service, χ_2 is a service time of a demand being on service at this time moment. Generally χ_1 and χ_2 random variables are dependent. But, if we fix the number of demands in the system ($\eta = n, n = 1, 2, \ldots$) and the time from the beginning of the service (when $\eta > 0$) to τ moment ($\xi^* = y, y > 0$), then χ_1 and χ_2 be independent. Denote as $p_n(y)dy = \mathbf{P}\{\eta = n, \xi^* \in [y; y+dy)\}$ the probability that at arbitrary time moment τ (in stationary mode) there are n demands in the system ($n > 0$) and the time interval from the beginning of the service and the τ moment lies in $[y; y+dy)$ interval. Then the distribution function $R(t)$ of χ random variable is

$$R(t) = \mathbf{P}\{\chi < t\} = p_0 + \sum_{n=1}^{\infty} \int_0^{\infty} \mathbf{P}\{\chi < t | \eta = n, \xi^* = y\} p_n(y)dy. \quad (1)$$

Let us introduce the next conditional distribution functions

$$R(t|\eta = n, \xi^* = y) = \mathbf{P}\{\chi < t | \eta = n, \xi^* = y\},$$

$$R_1(t|\eta=n)=\mathbf{P}\{\chi_1<t|\eta=n\}, R_2(t|\xi^*=y)=\mathbf{P}\{\chi_2<t|\xi^*=y\}$$

and corresponding Laplace-Stieltjes transforms $r(q|\eta=n,\xi^*=y)$, $r_1(q|\eta=n)$ and $r_2(q|\xi^*=y)$. Passing in (1) to the Laplace-Stieltjes transform, from the fact of independence of χ_1 and χ_2 when above conditions take place we have

$$r(q)=1-\rho+\sum_{n=1}^{\infty}\int_0^{\infty} r(q|\eta=n,\xi^*=y)p_n(y)dy=$$

$$=1-\rho+\sum_{n=1}^{\infty}\int_0^{\infty} r_1(q|\eta=n)r_2(q|\xi^*=y)p_n(y)dy. \quad (2)$$

As far as the number of waiting demands is $n-1$ when $\eta=n$, then in our conditions the χ_1 random variable has an Erlang distribution of $n-1$–th order with a parameter $\mu=f/c$. Hence, $r_1(q|\eta=n)=\mu^{n-1}/((\mu+q)^{n-1})$. Since service time ξ has an exponential distribution, then $R_2(t|\xi^*=y)=1-e^{-\mu(t-y)}$, from which $r_2(q|\xi^*=y)=e^{-qy}\mu/(\mu+q)$.

Then the relationship (2) may be presented as follows

$$r(q)=1-\rho+\sum_{n=1}^{\infty}\int_0^{\infty} e^{-qy}\frac{\mu^n}{(\mu+q)^n}p_n(y)dy=$$

$$=1-\rho+\int_0^{\infty} e^{-qy}\sum_{n=1}^{\infty}\frac{\mu^n}{(\mu+q)^n}p_n(y)dy. \quad (3)$$

Let us introduce the generating function $P(z,y)=\sum_{n=1}^{\infty}z^n p_n(y)$. Denote as $p(z,q)$ the Laplace-Stieltjes transform of this function on y. Taking into account the above notations, the relationship (3) may be rewritten as follows

$$r(q)=1-\rho+p\left(\frac{\mu}{\mu+q},q\right). \quad (4)$$

So solution of the problem comes down to determination of $p(z,q)$ function. Let us determine the probability $p_n(y)dy$.

$$\mathbf{P}\{\eta=n,\xi^*\in[y;y+dy)\}=\mathbf{P}\{\eta>0\}\mathbf{P}\{\eta=n,\xi^*\in[y;y+dy)|\eta>0\}=$$

$$=\rho\mathbf{P}\{\eta=n|\xi^*=y\}\mathbf{P}\{\xi^*\in[y;y+dy)|\eta>0\}=\rho\mathbf{P}\{\eta=n|\xi^*=y\}\mu e^{-\mu y}dy,$$

i.e.

$$p_n(y)=\rho\mu e^{-\mu y}\mathbf{P}\{\eta=n|\xi^*=y\}. \quad (5)$$

Let $\Pi_k(t)$ be the probability of entering of k demands $(k = 0, 1, \ldots)$ of recurrent entrance flow to the system in time interval $[0; t)$. For our system the distribution function of time interval between neighboring entering moments is $A(t)$. The event $\{\eta = n | \xi^* = y\} (n > 0)$ takes place at an arbitrary time moment τ in the case of

1) the demand being on service at the time moment τ came to the system when there were no demands in it (the probability of this event is equal to π_0), and then during y time period $n-1$ demands came to the system (the probability of this is $\Pi_{n-1}(y)$), or

2) the demand being on service at time moment τ came to the system when there were $k (k \geq 1)$ demands in it (the probability of this event is equal to π_k), then during some time period t $[0, \infty) k$ demands completed the service (as far as the service time has an exponential distribution, the probability of this event during time interval $[t, t+dt)$ is equal to $\frac{\mu(\mu t)^{k-1} e^{-\mu t}}{(k-1)!} dt$), and during the time period $t + y$ from the moment of entering the demand being on service at τ moment $n-1$ demands came to the system (the probability of this event is equal to $\Pi_{n-1}(t+y)$).

As a result we have, taking in consideration the known relations for π_k,

$$\mathbf{P}\{\eta = n | \xi^* = y\} = \pi_0 \Pi_{n-1}(y) + \sum_{k=1}^{\infty} \pi_k \int_0^{\infty} \frac{\mu(\mu t)^{k-1}}{(k-1)!} e^{-\mu t} \Pi_{n-1}(t+y) dt =$$

$$= (1-\kappa) \left[\Pi_{n-1}(y) + \kappa\mu \int_0^{\infty} e^{-\mu t} \sum_{k=1}^{\infty} \frac{(\mu\kappa t)^{k-1}}{(k-1)!} \Pi_{n-1}(t+y) dt \right] =$$

$$= (1-\kappa) \left[\Pi_{n-1}(y) + \kappa\mu \int_0^{\infty} e^{-(1-\kappa)\mu t} \Pi_{n-1}(t+y) dt \right].$$

So, as it follows from (5),

$$p_n(y) = \rho\mu(1-\kappa) e^{-\mu y} \left[\Pi_{n-1}(y) + \kappa\mu \int_0^{\infty} e^{-(1-\kappa)\mu t} \Pi_{n-1}(t+y) dt \right]. \quad (6)$$

Let us introduce the generating function $\Pi(z,t) = \sum_{k=0}^{\infty} z^k \Pi_k(t)$ and find $P(z,y)$. As it follows from (6),

$$P(z,y) =$$

$$= \rho\mu(1-\kappa) e^{-\mu y} z \left[\sum_{n=1}^{\infty} z^{n-1} \Pi_{n-1}(y) + \kappa\mu \int_0^{\infty} e^{-(1-\kappa)\mu t} \sum_{n=1}^{\infty} z^{n-1} \Pi_{n-1}(t+y) dt \right] =$$

$$= \rho\mu(1-\kappa)e^{-\mu y}z\left[\Pi(z,y) + \kappa\mu\int\limits_0^\infty e^{-(1-\kappa)\mu t}\Pi(z,t+y)dt\right].$$

Denote as $\pi(z,q)$ the Laplace transform on t of $\Pi(z,t)$ function and find $p(z,q)$.

$$p(z,q) == \rho\mu(1-\kappa)z\left\{\int\limits_0^\infty e^{-(q+\mu)y}\Pi(z,y)dy + \right.$$

$$\left. +\kappa\mu\int\limits_0^\infty e^{-(q+\mu)y}\left[\int\limits_0^\infty e^{-(1-\kappa)\mu t}\Pi(z,t+y)dt\right]dy\right\} =$$

$$\rho\mu(1-\kappa)z\left\{\pi(z,q+\mu) + \kappa\mu\int\limits_0^\infty e^{-(q+\mu)y}\left[\int\limits_y^\infty e^{-(1-\kappa)\mu(t-y)}\Pi(z,t)dt\right]dy\right\} =$$

$$\rho\mu(1-\kappa)z\left\{\pi(z,q+\mu) + \kappa\mu\int\limits_0^\infty e^{-(q+\kappa\mu)y}\left[\int\limits_y^\infty e^{-(1-\kappa)\mu t}\Pi(z,t)dt\right]dy\right\} =$$

$$\rho\mu(1-\kappa)z\left\{\pi(z,q+\mu) + \kappa\mu\int\limits_0^\infty e^{-(1-\kappa)\mu t}\Pi(z,t)\left[\int\limits_0^t e^{-(q+\kappa\mu)y}dy\right]dt\right\}.$$

From the latest relation we have after some simple transformations

$$p(z,q) = \frac{\rho\mu(1-\kappa)z}{q+\mu\kappa}[q\pi(z,q+\mu) + \kappa\mu\pi(z,\mu-\mu\kappa)]. \tag{7}$$

It is known [5], that $\pi(z,q) = \frac{1-\alpha(q)}{q(1-z\alpha(q))}$. From this relation and from (4), (7) we have after some simple transformations

$$r(q) = 1-\rho+\frac{\rho\mu^2(1-\kappa)}{q+\mu\kappa}\left[\frac{\kappa}{q+\mu-\mu\kappa} + \frac{q(1-\alpha(q+\mu))}{(q+\mu)(q+\mu-\mu\alpha(q+\mu))}\right]. \tag{8}$$

From (8) the next evident relation follows

$$\delta(s) = 1-\rho+\frac{\rho f^2(1-\kappa)}{s+f\kappa}\left[\frac{\kappa}{s+f-f\kappa} + \right.$$

$$\left. + \frac{s[1-\alpha((s+f)/c)]}{(s+f)(s+f-f\alpha((s+f)/c))}\right]. \tag{9}$$

We can determine moments of any order of σ random variable (if they exist) from (9) relation, and in some cases we can determine the relation for the distribution function $D(x)$ of σ random variable. For example, for two the first moments δ_1 and δ_2 of σ the next relations take place:

$$\delta_1 = -\delta'(0) = \frac{\rho(2-\kappa)}{f(1-\kappa)}, \tag{10}$$

$$\delta_2 = \delta''(0) = \frac{2\rho}{\kappa f^2}\left[\frac{\kappa^2+2(1-\kappa)}{(1-\kappa)^2} - \frac{(1-\kappa)(2-\alpha(f/c))}{1-\alpha(f/c)}\right]. \tag{11}$$

Note, that (10) relation can be obtained from the analog of J.Little's formula for queueing systems with random length demands [3]. The (10) and (11) relations may be used for approximation of summarized volume distribution function to estimate characteristics of losing for queues with restricted summarized volume [1].

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