Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka VI, Częstochowa 1999

On some consequences of universal sets

Tomáš Zdráhal

1. *Introduction and summary of results.* Given a set A of real numbers. If for every real sequence $\{\lambda_n\}$ converging to 0 there is a number ξ such that $\xi + \lambda_n \in A$ for all sufficiently large n, we say A is universal set and write $A \in \mathcal{U}$.

Among other results H. Kestelman [1] has proved that if $A \in \mathcal{U}$ then the „distances of A" (i. e. the set $D(A) = \{|x-y| \in \mathbb{R}\colon x, y \in A\}$) contain a ball, we write $A \in \mathcal{B}$. One of the well known Steinhaus' theorem says that every set of positive Lebesque measure belongs to $\mathcal{B}$.

The main aim of this paper is to generalize the notion of universal set and to gain generalization of its consequence.

2. *Generalization.* Suppose that to every ω belonging to a metric space Ω, there is a certain transformation T_ω transforming a Lebesque measurable set in N-dimensional Euclidean space $\mathbb{R}_N$ into a Lebesque measurable set in $\mathbb{R}_N$. Following M. Pal and S. Panda [3] we shall consider such families of transformations satisfying the following conditions.

(i) There exists $\omega_0 \in \Omega$ such that for every sequence $\{\omega_n\}$ $(\omega_n \in \Omega)$ converging to ω_0 and for every compact set C in $\mathbb{R}_N$ the sequence $\{T_{\omega_n}(\xi)\}$ converges uniformly to ξ on C.

(ii) If E and F are measurable sets in $\mathbb{R}_N$ such that $E \subset F$, then for every $\omega \in \Omega$

$$T_\omega(E) \subset T_\omega(F).$$

(iii) For every sequence $\{\omega_n\}$ converging to ω_0 and for every measurable set E

$$\lim |T_{\omega_n}(E)| = |T_{\omega_0}(E)| = |E|,$$

where e.g. $|E|$ denotes the Lebesque measure of E.

Example 1. Put $\Omega = \mathbb{R}_1$ ($\mathbb{R}_1$ is supposed to be the metric space with Euclidean metric). If $E \in \mathcal{L}$ ($\mathcal{L}$ denotes the family of all Lebesque measurable subsets of the set $\mathbb{R}_1$) then let

$$T_\omega(E) = E + \omega.$$

Taking 0 as ω_0 one can check easily that properties (i) – (iii) are satisfied.

Example 2. Put $\Omega = \langle 0,1\rangle$ ($\langle 0,1\rangle$ is supposed to be the metric space with the Euclidean metric). If $E \in \mathcal{L}$, then let

$$T_\omega(E) = \omega E.$$

If we put $\omega_0 = 1$ then properties (i) – (iii) are again satisfied.

In what follows we shall need the following lemma and its corollaries:

Lemma 1 *Let the sequence $\{T_{\omega_n}\}$ of transformations satisfies the condition (i) and let C be compact set of positive Lebesque measure in $\mathbb{R}_N$. Then for every $\varepsilon > 0$, there exists a positive integer n_0 such that for $n \geq n_0$*

$$|T_{\omega_n}(C) \cap C| > |T_{\omega_n}(C)| - \varepsilon.$$

Proof. Let $\varepsilon > 0$ and let U be an open set containing the set C such that

$$||U| - |C|| < \varepsilon.$$

Let δ be the distance between $R_N \setminus U$ (the complement of U) and C. Let $\xi \in C$. Then, on account of (i), for all sufficiently large n

$$|T_{\omega_n}(\xi) - \xi| < \min(\varepsilon, \delta).$$

So, $T_{\omega_n}(C) \subset U$ for all large n. Further, we have

$$T_{\omega_n}(C) \cap C = U \setminus \{[U \setminus T_{\omega_n}(C)] \cup [U \setminus C]\}.$$

Hence for all sufficiently large n

$$\begin{aligned} |T_{\omega_n}(C) \cap C| &\geq |U| - |U \setminus T_{\omega_n}(C)| - |U \setminus C| \\ &= |U| - |U| + |T_{\omega_n}(C)| - |U| + |C| \\ &= |T_{\omega_n}(C)| - |U| + |C| \\ &> |T_{\omega_n}(C)| - \varepsilon, \end{aligned}$$

what completes the proof.

Corollary 1 *If in additional the sequence $\{T_{\omega_n}\}$ also satisfies the condition (iii) then*

$$\lim_{n\to\infty} |T_{\omega_n} \cap C| = |C|.$$

Next corollary is the direct consequence of the foregoing Lemma and Corollary 1.

Corollary 2 *If the sequence $\{T_{\omega_n}\}$ of transformations satisfies the conditions (i) and (iii) and if C is a compact set in $\mathbb{R}_N$, then*

$$\lim_{n\to\infty} |C \setminus T_{\omega_n}(C)| = 0.$$

Theorem. *Let A_i, $i = 0, 1, \ldots$ be compact sets of positive Lebesque measure in $\mathbb{R}_N$ having a common point of density. Let there exists a point ω_0 in a metric space Ω such that for a sequence $\{\omega_n\}$ ($\omega_n \in \Omega$, $\omega_n \neq \omega_{n+1}$, $n = 1, 2, \ldots$) converging to ω_0 the sequence $\{T_{\omega_n}\}$ of transformations satisfies the conditions (i) – (iii).*

Then there exists a subsequence $\{\omega_{n_i}\}$ of the sequence $\{\omega_n\}$ such that

$$A_0 \cap \bigcap_{j=1}^{\infty} \left(\bigcap_{i=1}^{\infty} T_{\omega_{n_i}}(A_j) \right)$$

is a set of positive measure.

Proof. By density theorem there exists a closed ball B with the centre c, where c is a common point of density of the sets A_i, $i = 0, 1, \ldots$, such that

$$|B \cap A_i| > \left(1 - \frac{\varepsilon}{2^i}\right)|B|,$$

$i = 0, 1, \ldots$, where $0 < \varepsilon < \frac{1}{2}$.

Hence

$$\left| B \cap \left(\bigcap_{i=0}^{\infty} A_i \right) \right| \geq |B| - \sum_{i=0}^{\infty} |B \setminus A_i|$$

$$> |B| - \sum_{i=0}^{\infty} \frac{\varepsilon}{2^i}|B|$$

$$> 2|B|\left(\frac{1}{2} - \varepsilon\right) > 0.$$

Let for $j = 1, 2, \ldots$ and $n = 1, 2, \ldots$

$$M_n^j = \left| \left\{ A_0 \cap \left(\bigcap_{i=1}^{\infty} A_i \cap B \right) \right\} \setminus \{A_0 \cap T_{\omega_n}(A_j \cap B)\} \right|.$$

Then

$$M_n^j \leq \left| \left(\bigcap_{i=1}^{\infty} A_i \cap B \right) \setminus T_{\omega_n}(A_j \cap B) \right|.$$

In virtue of Corollary 2

$$\lim_{n\to\infty} \left| \left(\bigcap_{i=1}^{\infty} A_i \cap B \right) \setminus T_{\omega_n}(A_j \cap B) \right| = 0.$$

Hence for $j = 1, 2, \ldots$

$$\lim_{n\to\infty} M_n^j = 0.$$

Then there exists the sequence $\{n_i\}$ of positive integers with $n_1 < n_2 < \ldots$ such that for every $j = 1, 2, \ldots$

$$M_{n_i} < \frac{\varepsilon_j}{2^i},$$

$i = 1, 2, \ldots$, where $\varepsilon_j = \frac{\varepsilon'}{2^j}$, $0 < \varepsilon' < |A_0 \cap (\bigcap_{i=1}^{\infty} A_i \cap B)|$.
Let

$$S = A_0 \cap \left(\bigcap_{i=1}^{\infty} A_i \cap B\right) \cap \left\{[A_0 \cap T_{\omega_{n_1}}(A_1 \cap B)] \cap [A_0 \cap T_{\omega_{n_2}}(A_1 \cap B)] \cap \ldots\right\}$$

$$\cap \left\{[A_0 \cap T_{\omega_{n_1}}(A_2 \cap B)] \cap [A_0 \cap T_{\omega_{n_2}}(A_2 \cap B)] \cap \ldots\right\} \cap \ldots$$

$$= \left\{A_0 \cap \left(\bigcap_{i=1}^{\infty} A_i \cap B\right)\right\} \setminus \left[\left\{A_0 \cap \left(\bigcap_{i=1}^{\infty} A_i \cap B\right)\right\} \setminus \{A_0 \cap T_{\omega_{n_1}}(A_1 \cap B)\}\right.$$

$$\cup \left\{A_0 \cap \left(\bigcap_{i=1}^{\infty} A_i \cap B\right)\right\} \setminus \{A_0 \cap T_{\omega_{n_2}}(A_1 \cap B)\} \cup \ldots$$

$$\cup \left\{A_0 \cap \left(\bigcap_{i=1}^{\infty} A_i \cap B\right)\right\} \setminus \{A_0 \cap T_{\omega_{n_1}}(A_2 \cap B)\}$$

$$\left.\cup \left\{A_0 \cap \left(\bigcap_{i=1}^{\infty} A_i \cap B\right)\right\} \setminus \{A_0 \cap T_{\omega_{n_2}}(A_2 \cap B)\} \cup \ldots\right]$$

Hence

$$|S| > \left|A_0 \cap \left(\bigcap_{i=1}^{\infty} A_i \cap B\right)\right| - \left[\left|\left\{A_0 \cap \left(\bigcap_{i=1}^{\infty} A_i \cap B\right)\right\} \setminus \{A_0 \cap T_{\omega_{n_1}}(A_1 \cap B)\}\right|\right.$$

$$+ \left|\left\{A_0 \cap \left(\bigcap_{i=1}^{\infty} A_i \cap B\right)\right\} \setminus \{A_0 \cap T_{\omega_{n_2}}(A_1 \cap B)\}\right|$$

$$+ \ldots + \left|\left\{A_0 \cap \left(\bigcap_{i=1}^{\infty} A_i \cap B\right)\right\} \setminus \{A_0 \cap T_{\omega_{n_1}}(A_2 \cap B)\}\right|$$

$$\left.+ \left|\left\{A_0 \cap \left(\bigcap_{i=1}^{\infty} A_i \cap B\right)\right\} \setminus \{A_0 \cap T_{\omega_{n_2}}(A_2 \cap B)\}\right| + \ldots\right]$$

$$> \left|A_0 \cap \left(\bigcap_{i=1}^{\infty} A_i \cap B\right)\right| - \left(\frac{\varepsilon'}{2^1} + \frac{\varepsilon'}{2^2} + \cdots\right) > 0,$$

since $0 < \varepsilon' < |A_0 \cap (\bigcap_{i=1}^{\infty} A_i \cap B)|$.
By applying condition (ii) we obtain that

$$A_0 \cap \bigcap_{j=1}^{\infty} \left(\bigcap_{i=1}^{\infty} T_{\omega_{n_i}}(A_j) \right)$$

is a set of positive Lebesque measure in $\mathbf{R}_N$. This completes the proof.

References

[1] H. Kestelman, *The convergent sequence belonging to a set,* J. London Math. Soc. XXII (1947), 130 – 135.

[2] T. Neubrunn and T. Šalát, *Distance sets, Ratio sets and Certain transformations of sets of real numbers,* Čas. Pěst. Mat. 94 (1969), 381 – 393.

[3] M. Pal and S. Panda, *On properties of sets under certain transformations,* Glas. Mat. Vol. 30 (50) (1995), 181 – 192.

Tomáš Zdráhal
Department of Mathematics
J.E.Purkyně University
České mládeže 8
400 96 Ústí nad Labem, Czech Republic