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A Method of Axiomatic Rejection of Formulas in Propositional Logics

Piotr Borowik, Grzegorz Bryll

The concept of a rejected formula, or generally: the concept of a rejection system axiomatizing the set of the nontheorems of a logic was introduced by J. Łukasiewicz in connection with his research on Aristotle's syllogistic (Cf. Łukasiewicz 1951). Later the notion was carried over to the methodology of propositional calculi and gained a more general formal shape - (see, e.g. Słupecki, Bryll and Wybraniec - Skardowska 1971.).

In this paper we want to introduce more important results about Ł - dacidable logics with giving the right systems of rejected axioms.

Throughout the present paper, the symbol J will denote propositional languages. Any such a language will be formally identified with the pair (S, F), where S is the set of all propositional variables and of all formulas (sentences) generated from the set At of propositional variables by the connectives F; we assume that the connective of implication, denoted by the symbol C, always belongs to F.

Let us fix two sets of formulas: A and A^{-1}; we call A (resp. A^{-1}) the set of *recognized* (resp. *rejected*) *axioms.* Similarly, let us fix two sets of inference rules $\mathcal{R}$ and $\mathcal{R}^{-1}$; we call the elements of $\mathcal{R}$ the *rules of recognition* (or acceptance), and we call the elements of $\mathcal{R}^{-1}$ the *rules of rejection.* Such a quadruple $(A, A^{-1}, \mathcal{R}, \mathcal{R}^{-1})$ determines uniquely a *propositional calculus;* we shal use the symbol **L** to denote propositional calculi of such a kind. Further, by $\mathcal{T}$ we mean the set of all formulas consisted of the members of $\mathcal{A}$ and all formulas which are derivable from $\mathcal{A}$ by means of the rules of $\mathcal{R}$, i.e. $\mathcal{T} = \mathcal{A} \cup Cn_{\mathcal{R}}(\mathcal{A})$; similarly, we put $\mathcal{T}^{-1} = \mathcal{A}^{-1} \cup Cn_{\mathcal{R}^{-1}}(\mathcal{A})^{-1}$. Thus $\mathcal{T}$ is the *set of all recognized formulas of* **L**, and $\mathcal{T}^{-1}$ is the *set of all rejected formulas of* **L**.

We say that the propositional calculus **L** is *Ł - decidable* (or *decidable in the Łukasiewicz sense*) if we have:

(w_1) $\mathcal{T} \cap \mathcal{T}^{-1} = \emptyset$, (Ł - consistency),

(w_2) $\mathcal{T} \cup \mathcal{T}^{-1} = S$, (Ł - completeness),

Thus **L** is Ł - decidable iff every formula of **L** is either recognized or rejected but not both.

J. Słupecki observed (in Słupecki 1972) that any calculus is decidable in the usual sense provided the calculus is Ł - decidable and the sets $\mathcal{T}$ and $\mathcal{T}^{-1}$ are recursively enumerable. This is an immediate corollary to a well-known result from recursion theory.

We shall write $\dashv \alpha$ (resp. $\vdash \alpha$) to express the fact that the formula $\alpha \in S$ belongs to $\mathcal{T}^{-1}$ (i.e. α is rejected) or, respectively, that α belongs to $\mathcal{T}$ (i.e. α is recognized) in a given calculus **L**.

Łukasiewicz considered, in the case of classical propositional logic, the - two rules of recognition which are defined by the following schemata: Let α be any propositional formula. The inscription $\dashv\alpha$ and $\vdash\alpha$ denote respectively that formula α is rejected and recognized in a given calculus.

$$(r_1): \quad \frac{\vdash C\alpha\beta \qquad \vdash \alpha}{\vdash \beta} \quad \text{(detachement)}$$

$$(r_2): \quad \frac{\vdash \alpha \text{ and } \beta \in Sub(\{\alpha\})}{\vdash \beta} \quad \text{(substitution)},$$

and the two rules of rejection which are defined by the following schemata:

$$(r_1^{-1}): \quad \frac{\vdash C\alpha\beta \qquad \dashv \beta}{\dashv \alpha} \quad \text{(reverse detachement)}$$

$$(r_2^{-1}): \quad \frac{\dashv \beta \text{ and } \beta \in Sub(\{\alpha\})}{\dashv \alpha} \quad \text{(reverse substitution)};$$

$Sub(\{\alpha\})$ is the set of all substitution instances of the formula α.

Let us assume that $\mathfrak{M} = (U, V, \mathbf{f})$, where $\emptyset \neq V \subset U$, is a logical matrix which is adequate for **L**, i.e. $E(\mathfrak{M}) = \mathcal{T}$ (the set $E(\mathfrak{M})$ of all tautologies of $\mathfrak{M}$ is identical with $\mathcal{T}$). Now if a system of rejected axioms is chosen in such a manner that

$$\mathcal{A}^{-1} \subseteq S - E(\mathfrak{M})$$

and the rules from $\mathcal{R}^{-1}$ preserve nontautologies of $\mathfrak{M}$, then

$$\mathcal{T}^{-1} \subseteq S - E(\mathfrak{M}).$$

In that case to prove that the calculus **L** is Ł-decidable it is enough to show

(w_3) $\quad S - \mathcal{T} \subseteq \mathcal{T}^{-1}$

holds.

We say that the constant (element) a of U is definable in **L** if there is a formula $\alpha \in S$ such that $h(\alpha) = a$ for every homomorphism (valuation) h of the language J to the algebra $(U, \mathbf{f})$. Any formula which defines a will be denoted by the symbol φ_a.

Theorem 1 (Bryll 1996). Let a propositional calculus **L** based on the set of rules $\mathcal{R} = \{r_1, r_2\}$ and $\mathcal{R}^{-1} = \{r_1^{-1}, r_2^{-2}\}$ have an adequate logical matrix $(U, V, \mathbf{f})$.[1]

If, moreover, **L** satisfies the following three conditions:

(w_4) $Cpp \in \mathcal{T}$,

(w_5) all elements of U are definable in **L**,

(w_6) $\dashv \varphi_a$ for all $a \in U - V$,

then **L** is Ł-decidable.

Proof. We shall show that under the assumptions (w_4) – (w_6) the following condition

$$S - \mathcal{T} \subseteq \mathcal{T}^{-1}$$

is satisfied.

To prove this, let $\alpha \in S - T$. Since $T = E(\mathfrak{M})$, we have $\alpha \notin E(\mathfrak{M})$. Hence there is a homomorphism $h_o : S \to U$ and an element $a_o \in U - V$ such that $h(\alpha) = a_o$. Let $p_1, \ldots, p_m$ be all the propositional variables occuring in α, i.e. $\alpha = \alpha(p_1, \ldots, p_m)$. Let $\alpha^* = \alpha(p_1/\varphi_{a_1}, p_2/\varphi_{a_2} \ldots, p_m/\varphi_{a_m})$, i.e. α^* results from α by substituting the variable p_i by the formula φ_{a_i}, where $h_o(p_i) = a_i$ $(i = 1, \ldots, m; a_1, \ldots, a_m \in U)$. Then for every homomorphism $h : S \to U$ we have $h(\alpha^*(\varphi_{a_1}, \ldots, \varphi_{a_m}) = a_o$. From ($w_4$) it follows that $\vdash C\alpha^*(\varphi_{a_1}, \ldots \varphi_{a_m})\varphi_{a_o}$, while from ($w_6$) it follows that $\dashv \varphi_{a_o}$. From this, marking use of r_1^{-1}, we get $\dashv \alpha^*$. Since $\alpha^* \in Sub(\alpha)$ we conclude by r_2^{-1} that $\dashv \alpha$, i.e. $\alpha \in T^{-1}$; and our proof is finished.

With the help of Theorem 1 one can prove that the following propositional calculi are Ł-decidable:

(1) for every $n \geq 2$ the so called definitionally complete (or full) n-valued calculus of Łukasiewicz with the primitive connectives of the Łukasiewicz implication C, the Łukasiewicz negation N and the Słupecki functor T,

(2) the four-valued modal logic of Łukasiewicz (Łukasiewicz 1953);

(3) the n-valued calculus of Sobociński with an implication and a negation as the only primitives connectives (Sobociński 1936).

It is worth noting that sometimes one can prove condition (w_3) without assuming that all constants of an adequate matrix for the calculus in question are definable in this calculus. Namely, it is enough sometimes to choose an appropriate finite set $\mathcal{A}^{-1}$ of rejected axioms. In this manner it has been proved on the sets of rules $\mathcal{R}$ and $\mathcal{R}^{-1}$, are Ł-decidable:

[1] The set U can have arbitrarily many elements.

(4) the pure implicational n-valued calculus of Łukasiewicz , augmented by either one of the two sets of rejected axioms $\mathcal{A}_1^{-1}$ and $\mathcal{A}_2^{-1}$, where
$\mathcal{A}_1^{-1} = \{C[Cp]^n q[Cp]^{n-1}q\}$,
$\mathcal{A}_2^{-1} = \{C[CCpq]^n q[CCpp]^{n-1}q\}$;
and $[C\alpha]^o\beta = \beta$, $[C\alpha]^{k+1}\beta = C\alpha[C\alpha]^k\beta$.

(5) the implication-negational n-valued calculus of Łukasiwicz, angmented by the set

$$\mathcal{A}^{-1} = \{C[Cp]^n NCpp[Cp]^{n-1}NCpp\}$$

of rejected axioms (Bryll, Maduch 1968);

(6) the three-valued *nonsense-logic* of Piróg-Rzepecka (see Piróg-Rzepecka 1977), augmented by one rejected axiom $\dashv NKpNp$;

(7) certain fragmentary n-valued calculi of Słupecki, augmented by the following rejected axioms:

(a) $\dashv [Cp]^o p, \dashv [Cp]^1 p, \ldots, [Cp]^{\frac{n}{2}-1}p$ if n is even *or*

(b) $\dashv [Cp]p, \dashv [Cp]^-1p, \ldots, [Cp]^{\frac{n-3}{2}p}$ if p is odd;

(see Bryll, Hałkowska 1986).

There are, however, certain propositional calculi based on the original Łukasiewicz rules of rejection r_1^{-1} and r_2^{-1} for which no finite sets of rejected axioms exist. To discuss this problem, let is take any set $L, (L \subset S)$ and define two (consequences) consequence operators, C_L and C_L^* by the following conditions:

(C_L) $\alpha \in C_L(X)$ iff there is a squence of formulas $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

(a) $\alpha_n = \alpha$,

(b) for every $k \leq n : \alpha_k \in X$ or there is $i < k$ such that $C\alpha_i\alpha_k \in L$ or there is $j < k$ and a substitution (endomorphism) ε of the language J such that $\alpha_k = \varepsilon\alpha_j$.

(C_L^*) $\alpha \in C_L^*(X)$ iff there is a sequence of formulas $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

(a) $\alpha_n = \alpha$,

(b) for every $k \leq n : \alpha_k \in X$ or there is $i < k$ such that $C\alpha_k\alpha_i \in L$ or there is $j < k$ and a substitution (endomorphism) ε of the language J such that $a_j = \varepsilon a_k$.

The conquences operations C_L and C_L^* determine the corsesponding closure spaces

$$C_L - \text{Syst} \quad = \{X \subseteq S : C_L(X) = X\},$$

and

$$C_L^* - \text{Syst} \quad = \{X \subseteq S : C_L^*(X) = X\}.$$

By a *directed base* for systemu $L, L \subseteq S$, we mean any family $\mathbb{R} \subseteq 2^S$ which satisfies the following conditions (see Maduch 1973):

(a) $\mathbb{R} \neq \emptyset$,
(b) $\cap \mathbb{R} = L$,
(c) $X \in \mathbb{R} \Rightarrow (X \in C_L - \text{Syst} \;\&\; L \subseteq X)$,
(d) $X, Y \in \mathbb{R} \Rightarrow \exists_{Z \in \mathbb{R}}(Z \subseteq X \cap Y)$,

Under the above notation we have

Lemma 1.

(i) $X \in C_L - \text{Syst} \iff X' \in C_L^* - \text{Syst}$;

(ii) $C_L^*(X) = L' \forall_Y (Y \cap X = \emptyset \wedge L \subseteq Y \wedge Y \in C_L - \text{Syst} \Rightarrow Y = L)$

Concerning the non – existence of finite systems of rejected axioms for a logic we have the following result which is due to M. Maduch (see Maduch 1973):

Theorem 2.

If there is a directed base for a given propositional logic then no finite set of rejected axioms can form a complete rejected base together with the nules $\overset{-1}{r_1}$, $\overset{-1}{r_2}$.

Proof.

Let L be a logic in question, and let $\mathbb{R}$ be a directed base for L. Let us assume *a contrario* that Y_1 is a finite and complete set of rejected axioms for L. Hence we have

(i) $Y_1 \subseteq S - L$,

(ii) $C_L^*(Y) = S - L$.

By (ii), the set Y_1 is nonempty since $C_L^*(\emptyset) = \emptyset$. Let $Y_1 = \{\beta_1, \ldots, \beta_m\}$. From (i) it follows that $\beta_i \notin L$ for $i = 1, \ldots, m$ and $\beta_i \notin \cap\mathbb{R}$ for $\cap\mathbb{R} = L$. Hence for every $i = 1, \ldots, m$ there is $Z_i \in \mathbb{R}$ such that $\beta_i \notin Z_i$. This and our definition of a directed base (condition (d)) imply the existence of such a set $Z_o \in \mathbb{R}$ that $Z_o \subseteq Z_1 \cap \ldots \cap Z_m$. Obviously $Z_o \in C_L$–Syst and $L \subset Z_o$. Moreover, $Z_o \cap Y_1 = \emptyset$. This, together with (b) and Lemma 1(ii), gives $Z_o = L$ which is a contradiction. ■

It has been proved that several calculi do not possess a finite set of rejected axioms when we restrict ourselves to the standard rejection rules

of Łukasiewicz.
For example, the following calculi are of this kind:

(1) the implication-negational $\aleph_o$ -valued calculus of Łukasiewicz (see Gniazdowski 1973);

(2) the intuitionistic propositional calculus (see Maduch 1973);

(3) the modal system $S5$ of Lewis (see Bryll, Słupecki 1973).

If we are confronted with the problem of Ł-decidability of a calculus which have a directed base, we must look for new rules of rejection in order to possibly solve the problem.

The concept of Ł-docidability nay also be applied to the so called *invariant* propositional calculi. (In invariant formalizations of calculi neither the rule of subilitition r_2 non the rule of inverse sublitution r_2- are allowed.) But in this case one must select a recursive (possibly infinite) set of rejected axioms. A fuller discussion of the problem is contained in Bryll 1996. It turned out that the following five invariant calculi have recussive sets of the rejected axioms:

(1) the two valued implicational-negational sentential calculus and its pure implicational fragment;

(2) the three-valued implicational calculus of Łukasiewicz (Bryll, Sochacki 1998);

(3) certain theree-valued invariant systems of ,,nonsense-logics" (Zbrzezny 1990);

(4) the n-valued definitionally complete calculus of Łukasiewicz (Bryll, Sochacki 1995);

(5) the n-valued implicational - negational calculus of Łukasiewicz with the n being a prime natural number (Bryll, Sochacki 1995).

We do not know whether theore are recursive sets of rejected axioms for the invariant n-valued implicational-negational calculi of Łukasiewicz for odd n's, and for the pure implicatinal n-valued calculi of Łukasiewicz if $n > 3$.

Last but not least, the method of axiomatic rejection can be applied to other logical systems, e.g. certain first-order calculi or syllogistic of Aristotle etc.

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Piotr Borowik
Wyższa Szkoła Pedagogiczna
42-200 Częstochowa
ul. Jasnogórska 64

Grzegorz Bryll
Uniwersytet Opolski
45-052 Opole
ul. Oleska 48