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INVARIANT MEASURE ON THE σ – FIELD OF CHRISTENSEN MEASURABLE SETS

We shall consider translation invariant measures defined on the family of Christensen measurable sets in an Abelian Polish group which is not locally compact.

Definition 1. Let $(X, +)$ be a group (not necessarily Abelian) and let $\mu: \mathcal{M} \to [0, \infty]$ be a measure defined on a σ – field of subsets of a set X. We say that μ is invariant with respect to right (left) translations (briefly: right invariant, left invariant, respectively) if and only if $E+x \in \mathcal{M}$ $(x+E \in \mathcal{M})$ and $\mu(E+x)=\mu(E)$ $(\mu(x+E)=\mu(E))$ for every set $E \in \mathcal{M}$, and for every element $x \in X$. A measure μ is said to be invariant, if it is right and left invariant simultaneously.

In what follows, we denote the σ – field of Borel sets in a topological space X by $\mathcal{B}(X)$.

Definition 2. Let $(X, +)$ be an Abelian Polish group. For every probability measure $\mu: \mathcal{B}(X) \to [0, 1]$ we define the σ – field $\mathcal{M}_\mu$ in the following way:

$\mathcal{M}_\mu := \{A \subset X : A = B \cup C,\ C \subset D,\ B, D \in \mathcal{B}(X) \text{ and } \mu(D)=0\}$ and put:

$\mathcal{M} := \bigcap\{\mathcal{M}_\mu : \mu \text{ is a probability measure on } \mathcal{B}(X)\}$,

$M := \{\mu : \mu \text{ is a probability measure on } \mathcal{M}\}$,

$\mathcal{H}_0 := \{A \in \mathcal{M}$: there exists a measure $\mu \in M$ such that $\mu(A+x)=0$ for every $x \in X\}$,

$\mathcal{C}_0 := \{A \subset X : A \subset B \text{ and } B \in \mathcal{H}_0\}$,

$\mathcal{C} := \{A \subset X : A = B \cup C, B \in \mathcal{M}, C \in \mathcal{C}_0\}$.

The family $\mathcal{C}_0$ is said to be the family of Christensen zero sets in the group X, whereas the family $\mathcal{C}$ is called the family of Christensen measurable sets in X.

We proceed with the following lemma proved in [3] (cf. also [1]).

Lemma. The family $\mathcal{C}$ of all Christensen measurable sets in an Abelain Polish group $(X, +)$ forms a σ – field of subsets of X. Moreover, $\mathcal{C}$ is translation invariant whereas the family $\mathcal{C}_0$ of all Christensen zero sets in X yields a proper σ – ideal, i.e.

(i) $X \notin \mathcal{C}_0$;

(ii) $\{A_n : n \in N\} \subset \mathcal{C}_0$ implies $\bigcup_{n \in N} A_n \in \mathcal{C}_0$;

(iii) $A \in \mathcal{C}_0$ and $B \subset A$ implies $B \in \mathcal{C}_0$.

Finally, for any $x \in X$ and any $A \in \mathcal{C}_0$ one has $A + x \in \mathcal{C}_0$.

It is well known (see [4], Chapter IV, §15, (15.8)), that in every locally compact topological group $(X, +)$ there exists exactly one (up to a multiplicative constant) measure $h : \mathcal{B}(X) \to [0, \infty]$, invariant with respect to right (left) translations and such that $h(C) < \infty$ for every compact set $C \subset X$ and $h(U) > 0$ for every non-empty open set $U \subset X$. This measure is said to be right (left) Haar measure, or simply Haar measure in case of an Abelian group. The completion of h is termed complete Haar measure.

It is also known (see [1] and [3]), that if $(X, +)$ is an Abelian Polish locally compact group, then the family $\mathcal{C}$ of all Christensen measurable sets in X coincides with the class of all measurable sets in sense of complete Haar measure; moreover the family $\mathcal{C}_0$ of Christensen zero sets coincides with the class of all sets having (completed) Haar measure zero. Obviously, in this case there exist non-empty open sets having positive and finite Haar measure.

Bearing the above facts in mind, we may ask whether in an Abelian Polish group $(X, +)$ which is not locally compact there exists an invariant measure $\mu : \mathcal{C} \to [0, \infty]$ such that the family $\mathcal{C}_0$ coincides with the class of all μ-null sets. On account of Lemma one may easily check that the function $\mu : \mathcal{C} \to \{0, \infty\}$, such that $\mu(A) = 0$ for every set $A \in \mathcal{C}_0$, and $\mu(A) = \infty$ for every set $A \in \mathcal{C} \setminus \mathcal{C}_0$, yields a measure satysfying the conditions required.

Such an example is rather trivial. Nevertheless, as we shall see further on, there is no hope for more interesting examples. More precisely, we are going to show that, actually, in any not locally compact Abelian Polish topological group with a translation invariant metric there is no well-behaving translation invariant measure on the family of all Christensen measurable sets; in other words, every such invariant measure has rather pathological properties.

To see this, let us first introduce some further notations:

Definition 3. Let $(X, +)$ be a group. A metric $d : X \times X \to [0, \infty)$ is said to be right (left) translation invariant (briefly: right invariant, left invariant, respectively) if and only if $d(x, y) = d(x+z, y+z)$ $(d(x, y) = d(z+x, z+y))$ for all $x, y, z \in X$.

In case of an Abelian group we shall omit the adjective: right.

The neutral element in a given group X we denote by 0. Moreover, if $(X, +)$ is a topological group with a topology determined by a metric d, and $a \in X$, $r > 0$, then we put denotations:

$K(a,r):=\{x\in X: d(x,a)<r\}, K_r:=K(0,r)$,

$\bar{K}(a,r):=\{x\in X: d(x,a)\leq r\}, \bar{K}_r:=\bar{K}(0,r)$.

Definition 4 (see [2], Chapter 4, §3). A subset A of a metric space (X,d) is said to be dense in X up to a positive ε if for every point $x\in X$ there exists an element $a\in A$ such that $d(x,a)<\varepsilon$.

Definition 5 (see [2], Chapter 4, §3). A metric space (X,d) is said to be totally bounded, if for every $\varepsilon>0$ there exists a finite set $\{x_1,x_2,\dots,x_k\}$, which is dense in X up to ε.

Theorem. Let $(X,+)$ be a Polish topological group (not necessarily Abelian) with a right invariant metric d. Suppose that a space X is not locally compact. If $\mu:\mathcal{B}(X)\to[0,\infty]$ is a right invariant (left invariant) measure and μ does not vanish identically, then $\mu(K(x,s))=\infty$ for all $s>0$ and $x\in X$.

Proof. Assume for the indirect proof, that $\mu(K(a,r))<\infty$ for some $r>0$ and $a\in X$.
The right invariance of the metric d implies the equality $K(x,s)=K_s+x$ for all $x\in X$, $s>0$. Now, since a measure μ is right invariant, we get

(1) $\mu(K(x,s))=\mu(K_s)$ for all $x\in X$, $s>0$.

From conditions (1) it follows that $\mu(K_r)\neq\infty$.

Since a topological group $(X,+)$ is not locally compact, then for every $s>0$ the set $\mathrm{cl}\,K_s$ is not compact and, consequently, for every $s>0$, $\bar{K}_s$ is not compact, as well.

Let $t\in(0,\frac{1}{2}r)$ be arbitrarily fixed. Then $(\bar{K}_t, d|_{\bar{K}_t\times\bar{K}_t})$ is a non-compact complete metric space. Hence, in patricular, $\bar{K}_t$ is not totally bounded (see [2], Chapter 4, §3, TH.14). Therefore there exists a number $\varepsilon>0$, such that for every finite set $\{a_1,\dots,a_k\}$, $k\in N$, contained in $\bar{K}_t$ there exists an $x\in\bar{K}_t$ such that $d(a_i,x)\geq\varepsilon$ for $i\in\{1,\dots,k\}$. It follows that $\bar{K}_t\setminus\bigcup_{i=1}^{k}K(a_i,\varepsilon)\neq 0$ for every finite set $\{a_1,\dots,a_k\}$, $k\in N$, contained in $\bar{K}_t$. Obviously $\varepsilon\leq t$.

Let $x_1\in\bar{K}_t$ be fixed. Then $\bar{K}_t\setminus K(x_1,\varepsilon)\neq 0$ and we may choose an element x_2 in the set $\bar{K}_t\setminus K(x_1,\varepsilon)$. It is clear that $d(x_1,x_2)\geq\varepsilon$ and $\bar{K}_t\setminus(K(x_1,\varepsilon)\cup K(x_2,\varepsilon))\neq 0$. Assume, that for some $k\in N$ an element $x_k\in\bar{K}_t\setminus\bigcup_{i=1}^{k-1}K(x_i,\varepsilon)$ has been chosen. Then $d(x_k,x_i)\geq\varepsilon$ for $i\in\{1,\dots,k-1\}$ and $\bar{K}_t\setminus\bigcup_{i=1}^{k}K(x_i,\varepsilon)\neq 0$. Hence we may choose an element $x_{k+1}\in\bar{K}_t\setminus\bigcup_{i=1}^{k}K(x_i,\varepsilon)$. Obviously $d(x_{k+1},x_i)\geq\varepsilon$ for $i\in\{1,\dots,k\}$. Thus we were able to construct an infinite sequence $(x_k)_{k\in N}$ of elements of the set $\bar{K}_t$, such that $d(x_k,x_n)\geq\varepsilon$ for all $k\neq n$, $k,n\in N$.

Note that

(2) $K(x_k,\frac{\varepsilon}{2})\cap K(x_n,\frac{\varepsilon}{2})=0$ for all $k\neq n$, $k,n\in N$.

Moreover,

(3) $\bigcup_{k=1}^{\infty} K(x_k,\frac{\varepsilon}{2})\subset K_r$.

Indeed, if $x\in\bigcup_{k=1}^{\infty} K(x_k,\frac{\varepsilon}{2})$, then $d(x,x_j)<\frac{\varepsilon}{2}$ for some $j\in N$.

Hence $d(x,0)\leqslant d(x, x_j)+d(x_j,0)<\frac{\varepsilon}{2}+t\leqslant\frac{3}{2}t<\frac{3}{4}r<r$; therefore $x\in K_r$.

In addition,

(4) $\mu(K_s)>0$ for every $s>0$.

In fact, the equality $\mu(K_s)=0$ for some $s>0$, jointly with (1) and the separability of our metric space (X,d) imply $\mu(X)=0$, which is impossible because, by assumption, μ does vanish identically.

From conditions (2), (1) and (4) we infer that

$$\mu(\bigcup_{k=1}^{\infty} K(x_k,\tfrac{\varepsilon}{2}))=\sum_{k=1}^{\infty}\mu(K(x_k,\tfrac{\varepsilon}{2}))=\sum_{k=1}^{\infty}\mu(K_{\frac{\varepsilon}{2}})=\infty.$$

On the other hand, condition (3) implies

$$\mu(\bigcup_{k=1}^{\infty} K(x_k,\tfrac{\varepsilon}{2}))\leqslant\mu(K_r)<\infty,$$

a contradiction, which finishes this proof in case of a right invariant measure.

Note that if $\nu\colon \mathcal{B}(X)\to[0,\infty]$ is a left invariant measure, then the function $\mu\colon\mathcal{B}(X)\to[0,\infty]$ given by the formula: $\mu(E):=\nu(-E)$ for every set $E\in\mathcal{B}(X)$, yields a right invariant measure. In addition, $-K_s=K_s$ for every $s>0$. Therefore $\nu(K_s)=\infty$ for every $s>0$. Let $x\in X$ and $s>0$ be arbitrarily fixed. Then the set $-x+K(x,s)$ is open and $0\in -x+K(x,s)$. Consequently, $K_p\subset -x+K(x,s)$ for some $p>0$. We observe that

$$\nu(K(x,s))=\nu(-x+K(x,s))\geqslant\nu(K_p)=\infty.$$

Hence $\nu(K(x,s))=\infty$. This completes our proof.

Remark. Our Theorem remains valid for left invariant metric.

Corollary. Let $(X,+)$ be an Abelian Polish topological group with an invariant metric. If a space X is not locally compact and $\mu\colon\mathcal{C}\to[0,\infty]$ is a nontrvial invariant measure defined on the σ-field of all Christensen measurable sets in X, then $\mu(U)=\infty$ for every non-empty open set $U\subset X$.

Proof. From the construction of the σ-field $\mathcal{C}$ it follows that $\mathcal{B}(X)\subset\mathcal{C}$. Then $\mu|_{\mathcal{B}(X)}$ is an invariant measure which does not vanish identically. By virtue of our Theorem we have $\mu|_{\mathcal{B}(X)}(K(x,s))=\infty$ for all $x\in X$, $s>0$. Therefore $\mu(U)=\mu|_{\mathcal{B}(X)}(U)=\infty$ for every non-empty open set $U\subset X$. This finishes our proof.

In some not locally compact Abelian Polish topological groups there exist nontrivial invariant measures, whose families of all nullsets coincide with the collection of all Christensen zero sets and which assume positive and finite values for some open sets. Obviously, any such measure can not be defined on a σ-field of sets containing all Borel sets. We are not going to study such measures in the present paper. Here we note only, that such measure do exist; for instance, that is the case in the space X=C(T) of all continuous, real or complex functions defined on a compact, metrizable topological space T provided X is endowed with the uniform converegence metric.

Abstract. Assume (X, +) to be a Polish topological group (not necessarily Abelian) endowed with a translation invariant metric. It is shown that if X is not locally compact, then any nontrivial right (left) translation invariant measure μ on the σ-field of Borel sets of X (or, alternatively, on the σ-field of Christensen measurable sets in X) has the property that $\mu(U)=\infty$ for each nonempty open set $U\subset X$.

This proves that any not locally compact Abelian Polish group admits no well – behaving translation invariant measure on the family of all Christensen measurable sets.

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STRESZCZENIE

Zakładamy, że (X, +) jest polską grupą topologiczną (niekoniecznie abelową), wyposażoną w metrykę niezmienniczą ze względu na translację. Wykazujemy, że jeżeli X nie jest lokalnie zwarta, to każda nietrywialna miara μ niezmiennicza ze względu na prawe (lewe) translacje, określona na σ-ciele zbiorów borelowskich w X (lub, alternatywnie, na σ-ciele zbiorów mierzalnych w sensie Christensena w X) ma tę własność, że $\mu(U)=\infty$ dla każdego niepustego zbioru otwartego $U\subset X$.

To dowodzi, że każda taka abelowa grupa polska, która nie jest lokalnie zwarta, nie dopuszcza dobrze zachowującej się miary niezmienniczej ze względu na translacje, określonej na rodzinie wszystkich zbiorów mierzalnych w sensie Christensena.