Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka VII, Częstochowa 1999

The geometry of squares

Robert Sochacki, Leszek Jaworski

In this paper we define the notions which are equivalent to any primary notion of Hilbert's geometry (see [1]). We will use only the notion of square and relation of congruency of squares, as primary notions in this system.

Our system will be based on Leśniewski's Mereology. You will find a detailed lecture in [8]. The logical constans are: the equivalence symbol $\Leftrightarrow$, the negation symbol $\neg$, the implication symbol $\Rightarrow$, the disjunction symbol $\wedge$, the conjunction symbol $\vee$, the universal quantifer $\forall$, the existential symbol $\exists$. The variables bound by quantifier are placed directly after the sign of this quantifier, when variables are more than one, they shall be separated by commas.

We remind that the only primary notion of Leśniewski's mereology is relation $\leq$, sign $X \leq Y$ is read: the object X is a *proper or improper part* of the object Y.

The first three axioms of mereology are:

AI $X \leq X$.

AII $(X \leq Y \wedge Y \leq Z) \Rightarrow X \leq Z$.

AIII $(X \leq Y \wedge Y \leq X) \Rightarrow X = Y$.

The relation $\leq$ is a partial order, because it is reflexive (AI), transitive (AII) and asymmetric (AIII). To define the last axiom of mereology two relations will be defined first:

DI $X \clubsuit Y \Leftrightarrow \exists Z Z \leq X, Y$.

This relation means: the object X is *disjoint* from the object Y. Let the expression $f(X)$ mean: the object X *has the property* f. Then the new relation has the following form:

DII $X\delta f \Leftrightarrow \forall Z[f(Z) \Rightarrow Z \leq X] \wedge \forall Y[Y \leq X \Rightarrow \exists U(f(U) \wedge U \clubsuit Y)]$

Now we can introduce the last axiom of mereology:

AIV $\exists Y[f(Y) \Rightarrow \exists_1 X X\delta f$.

According to AIV, for each object, which has the property f there exists one and only one object X which is in the relation δ with the property f. This object will be denoted by the symbol $\sum_A f(A)$ and will be called the set of all those objects which have the property f, or the mereological sum. In accordance with DII the object X is therefore the set of all objects having the property f if and only if every object having that property is a part of Z and for every part of X there is an object not disjoint from it, which has that property.

Axiom AIV tells, that for each property which has at least one object there exists exactly one set determined by that property. As we see, the sets are determined by the property, but different as in The Set Theory. In the square geometry equivalent notions of point, line and plane are respectively: a notion of sphere with (the) given diameter d, notion of unlimited tunnel with set diameter too and unlimited layer with thickess size d. The expression $X \leq Y$ will be read: the object X is *coverable* by the object Y or the object Y *covers* the object X. In this system we are going to use the method introduced in [4]. We will use small letters $x, y, z, u, v, w, \ldots$ for squares, big letters $X, Y, Z, U, V, W, \ldots$ with adequate symbols we will use for defined objects.

Squares congruance x and y we denote $x \equiv y$. Let us remind that notion: the square and the relation of congruance are primary notions. The further relations defined in this paper are denoted by small Greek letters with an index. Let us assume now that the conjunction $a\varrho b$ and $a\varrho c$ we will note $a\varrho b, c$ (similarly for more variables), the note $a, b\varrho c$ will be equivalent to the conjunction of conditions $a\varrho c$ and $b\varrho c$ (similarly for more variables). In this paper the problem of axioms is omitted (the axioms of the theory of squares are pesented in [10]). We will use some qualifications (which will be put in quotation marks) for better understanding of the relations. It means that qualifications are not definited notions in this system and we will use them only for good interpretation of the introduced symbols.

1. Square cylinder

Before we define square cylinder we introduce a few auxiliary relations.

Definition 1
$x\varrho_1 y \Leftrightarrow \exists z x, y \leq z.$

Two squares are *complanar* if and only if they are a part of a square.

Definition 2
$x\varrho_2 y \Leftrightarrow \forall u \leq x \neg(u \leq y) \wedge \exists(b, c, d)[\neg(c \clubsuit d) \wedge c, d \leq b \wedge c \leq x \wedge d \leq y \wedge b \leq \sum_a (a = x \vee a = y)]$

In conformity with Definition 2 the square x is *externally tangent* to a square y by a "side". It holds if and only if for each square u, if u is a part of the square x then u is not a part of the square y and such squares b, c, d exist that the squares c and d are disjoint and the square b is a part of the mereological sum of x and y and the squares c, d are a part of the square b and the square c is a part of x and the square d is a part of y (fig. 1).

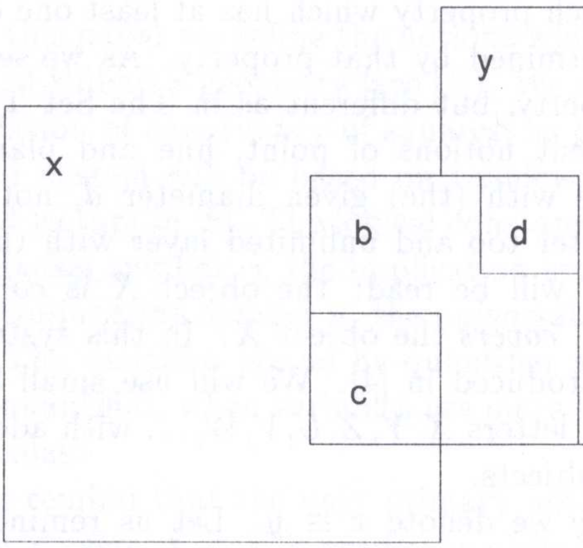

Fig. 1

Definition 3
$x\varrho_3 y \Leftrightarrow x \leq y \wedge \exists z z\varrho_2 x, y$

The relation ϱ_3 states, that the square x is *internally tangent* by a "side" to the square y (fig. 2).

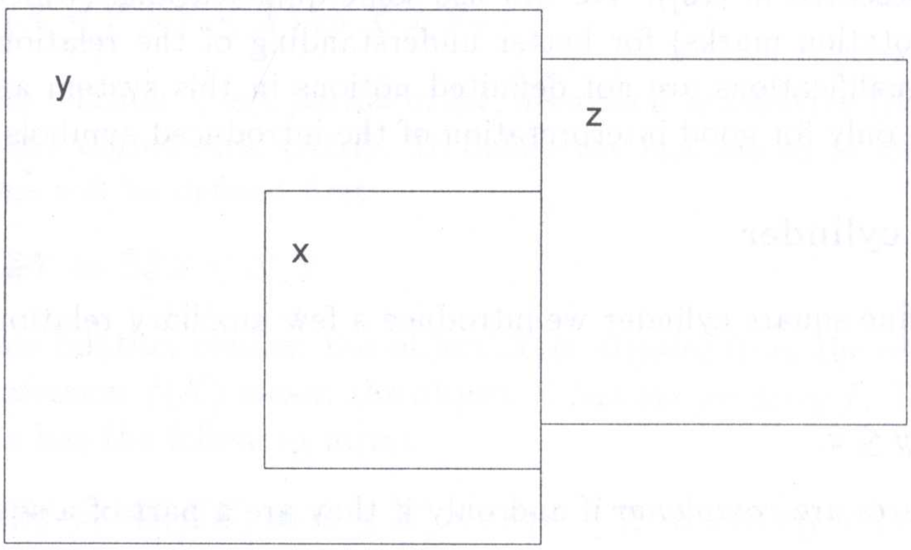

Fig. 2

Definition 4
$x\varrho_4 yz \Leftrightarrow x\varrho_2 y \wedge \forall u\varrho_3 y(x \leq u \Rightarrow u\varrho_2 z)$

The notation $x\varrho_4 yz$ is read: the square x is *externally tangent* to the squares y and z by the same "side" (fig. 3).

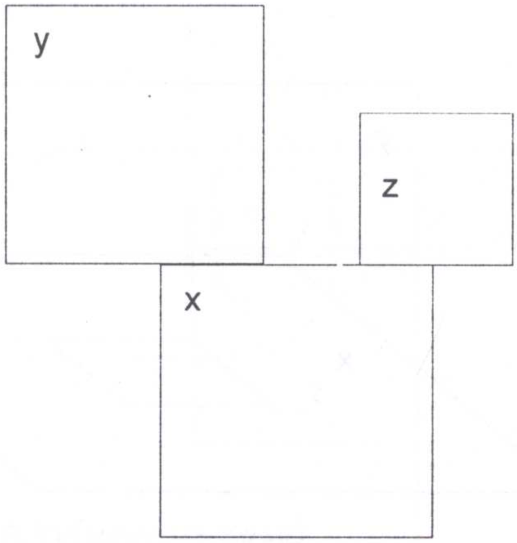

Fig. 3

Definition 5
$x\varrho_5 y \Leftrightarrow x\clubsuit y \wedge \exists z z\varrho_4 xy.$

The relation ϱ_5 describes the square x which *adheres internaly* with "side" to the square y, but the square x does not have to be a part of the square y, just like in Definition 3.

Definition 6
$x\varrho_6 y \Leftrightarrow x \leq y \wedge x \neq y \wedge \exists z \forall u \equiv z(u\varrho_2 x \Rightarrow u\varrho_3 y).$

The above expression $x\varrho_6 y$ is read: the square y is *symmetrical extension* of the square x (fig. 4).

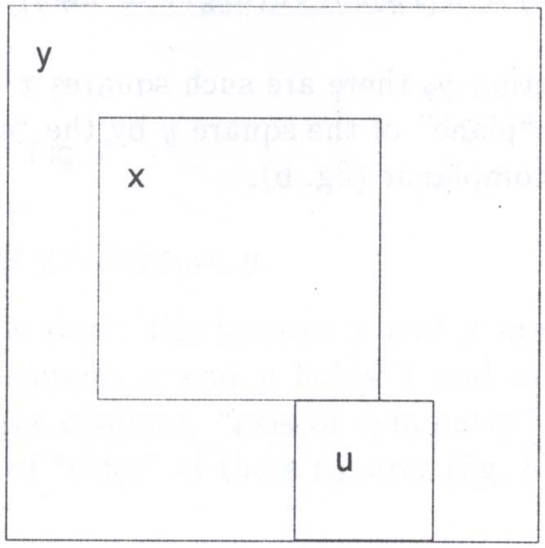

Fig. 4

Definition 7
$x\varrho_7 y \Leftrightarrow \forall z\varrho_6 x \forall u\varrho_6 y(z\clubsuit u)$.

The relation ϱ_7 between the squares x and y takes place if and only if they are *concentric* (fig. 5).

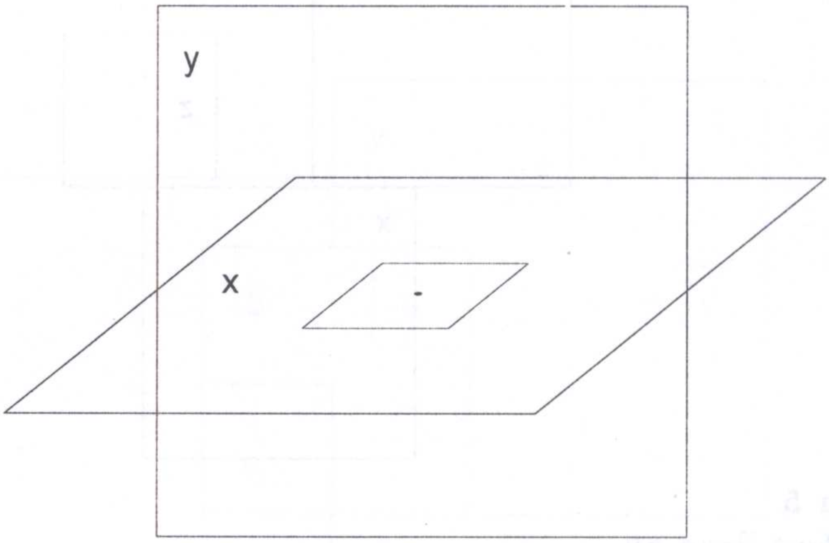

Fig. 5

Definition 8
$u\varrho_8 xy \Leftrightarrow x\varrho_2 y \wedge u \leq \sum_a(a = x \vee a = y) \wedge \exists(b,c)[b,c \leq u \wedge b \leq x \wedge c \leq y \wedge \neg(b\clubsuit c)]$.

The square u is in relation ϱ_8 with the squares x and y if and only if the squares x and y are externally tangent and square u is a part of the mereological sum of x and y and such squares b and c exist, that they are a part of u and b and c are a part of y. The square u will be called here a *connector* of x and y (fig. 1 where u is b).

Definition 9
$x\varrho_9 y \Leftrightarrow \neg(x\varrho_1 y) \wedge \exists z[x\varrho_2 z \wedge \forall u(u\varrho_8 xz \Rightarrow u\clubsuit y)]$.

In this relation ϱ_9 there are such squares x and y that the square x is tangent to the "plane" of the square y by the "side" or its part and x and y are not complanar (fig. 6).

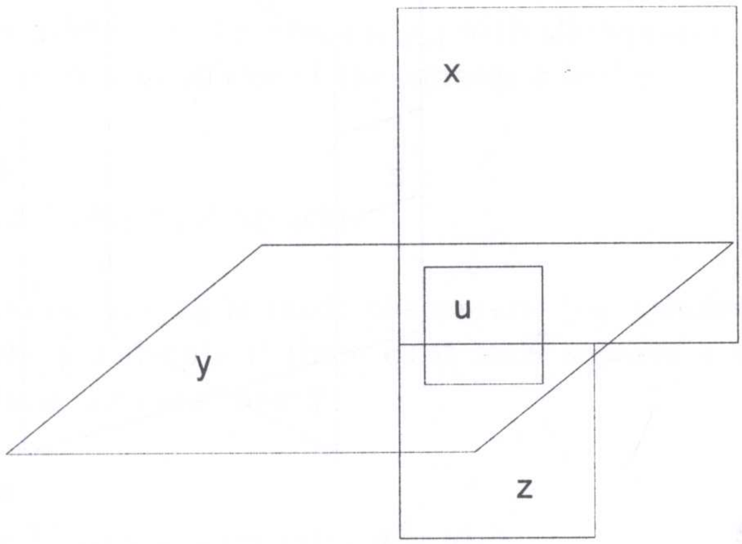

Fig. 6

Definition 10

$x\varrho_{10}y \Leftrightarrow \forall u(x\varrho_6 u \Rightarrow y\varrho_9 u) \wedge \forall v(y\varrho_6 v \Rightarrow x\varrho_9 v)$

The above definition describes the relation ϱ_{10} which concerns the squares x and y tangent themselves by "sides" or by "vertices", where the squares x and y have a couple of "parallel sides" (fig. 7).

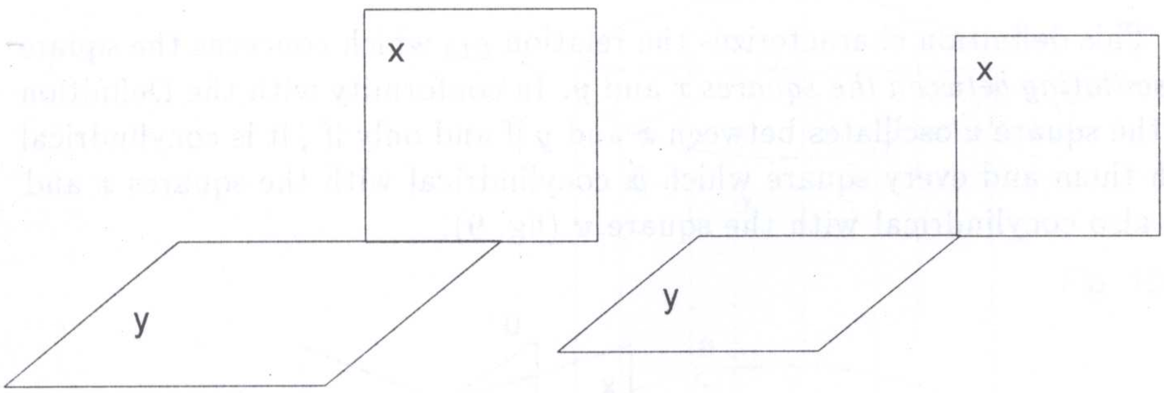

Fig. 7

Definition 11

$x\varrho_{11}y \Leftrightarrow x \neq y \wedge x\varrho_7 y \wedge x \equiv y \wedge \exists z z\varrho_{10}x, y.$

The expression $x\varrho_{11}y$ is read: the squares x and y are *cocylindrical.* The relation between the squares x and y holds if and only if x and y are congruent and they have common "axis of symmetry". This axis of symmetry includes centres of "sides" of these squares (fig. 8).

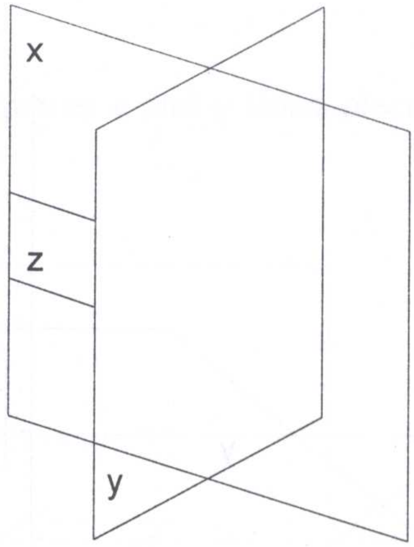

Fig. 8

Definition 12

$x\varrho_{12}y \Leftrightarrow \forall u\varrho_1 x \forall v\varrho_1 y \neg(u\clubsuit v)$.

Above were desribed *bi-level* squares. The squares x and y are bi-level if and only if every two squares u and v, complanar with the squares x and y respectively, are disjoint.

Definition 13

$u\varrho_{13}xy \Leftrightarrow x\varrho_{11}y \wedge u\varrho_{11}x, y \wedge \forall v(v\varrho_{11}x, y \Rightarrow u\varrho_{11}v)$.

This definition characterizes the relation ϱ_{13} which concerns the square u *oscillating between the squares* x and y. In conformity with the Definition 12, the square u oscillates between x and y if and only if , it is cocylindrical with them and every square which is cocylindrical with the squares x and y is also cocylindrical with the square u (fig. 9).

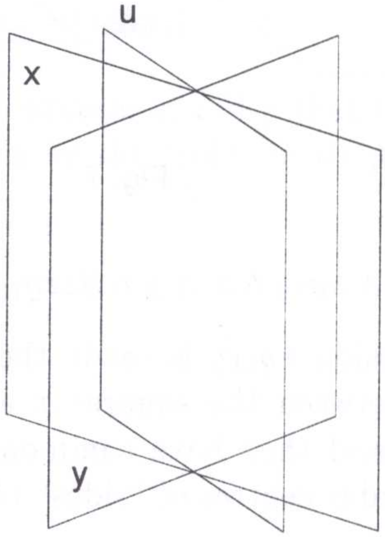

Fig. 9

Definition 14
$u\varrho_{14}xy \Leftrightarrow u\varrho_{13}xy \wedge \exists(a,b)[a \equiv b \wedge \neg(a\varrho_1 b) \wedge \neg(a\varrho_{12}b) \wedge a\varrho_{10}u \wedge b\varrho_{10}u]$.

The square u being in the relation ϱ_{14} with the squares x and y will be called here as *bisetrix oscillator* of the squares x and y.

Definition 15
$u\varrho_{15}xy \Leftrightarrow \exists(c,b)(c,b\varrho_{13}xy \wedge u\varrho_{14}cb)$.

The expression $u\varrho_{15}xy$ is read: the square u is *between* the square x and y. It holds if and only if there exist such squares c and b that the square u oscillates between them.

Definition 16
$Z\varrho_{16}xy \Leftrightarrow Z = \sum_a(a = x \vee a\varrho_{15}xy \vee a = y)$.

Now we have defined square cylinder Z. It is determined by two cocylindrical squares x and y. In conformity with the Definition 16 the square cylinder Z is the meorological sum of the squares x and y and all squares being between x and y. Let us assume that squares x and y generate cylinder Z (fig. 10).

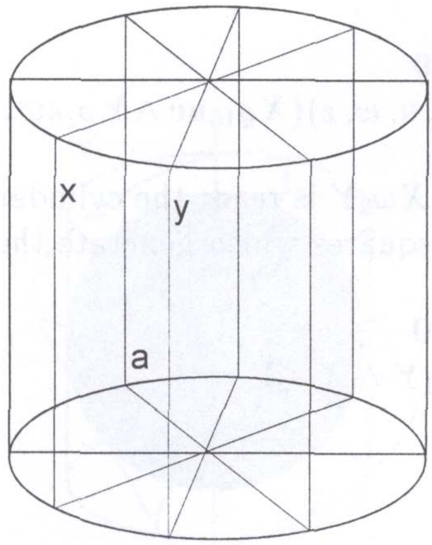

Fig. 10

Definition 17
$\boldsymbol{W}(Z) \Leftrightarrow \exists(x,y)Z\varrho_{16}xy$.

The symbol $\boldsymbol{W}(Z)$ means that, the object Z is a *square cylinder*. The object Z is a cylinder if and only if there exist squares x and y which generate the square cylinder Z.

2. Sphere

Before we define adequate equivalent symbol to the point we give some auxiliary relations.

Definition 18
$X\omega_1 Y \Leftrightarrow \exists(u,v,w,z)(X\varrho_{16}uv \wedge Y\varrho_{16}wz \wedge u\varrho_7 w).$

This expression is read: the cylinders X and Y are *concentric*. It means that squares which generate them are concentric (fig. 11).

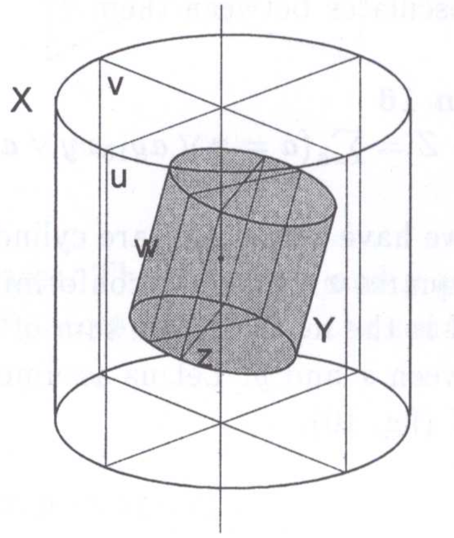

Fig. 11

Definition 19
$X\omega_2 Y \Leftrightarrow \exists(u,v,w,z)(X\varrho_{16}uv \wedge Y\varrho_{16}wz \wedge u \equiv w).$

The note $X\omega_2 Y$ is read: the cylinders X and Y are *congruent*. It holds if and only if squares which generate them are congruent too.

Definition 20
$X\omega_3 Y \Leftrightarrow X\omega_1 Y \wedge X\omega_2 Y.$

The relation ω_3 describes the congruent and concentric cylinders (fig. 12).

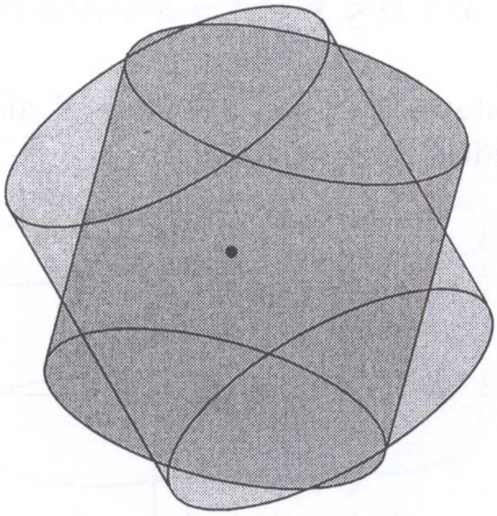

Fig. 12

Definition 21
$X\omega_4 Y \Leftrightarrow \boldsymbol{W}(Y) \wedge \forall U \omega_3 Y X \leq U \wedge \forall V[\forall Z \omega_3 Y (V \leq Z) \Rightarrow V \leq X]$.

The expression $X\omega_4 Y$ is read: X is a sphere *inscribed* into the cylinder Y, but X is the biggest object inscribed into this cylinder in sense of Definition 21 (fig. 13).

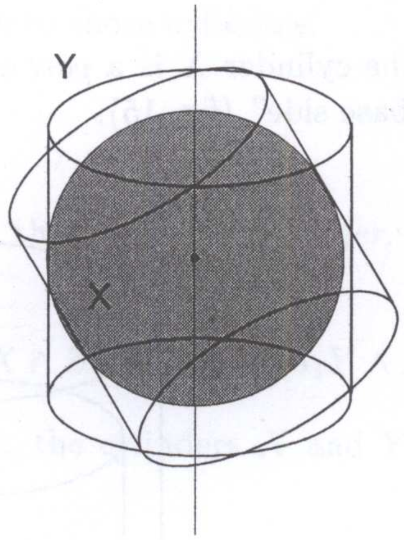

Fig. 13

Definition 22
$\boldsymbol{K}(X) \Leftrightarrow \exists Y X\omega_4 Y$

The symbol $\boldsymbol{K}(X)$ means, that the object X is a *sphere*.

3. Tunnel and layer

Definition 23
$X\alpha_1 Y \Leftrightarrow \boldsymbol{W}(X) \wedge \boldsymbol{W}(Y) \wedge \forall \boldsymbol{W}(U)[U \leq X \Rightarrow \neg(U \leq Y)] \wedge \exists(D,B,C)[B,C \leq D \wedge B \leq X \wedge C \leq X \wedge D \leq \sum_A (A = X \vee A = Y)]$

This relation states that two cylinders X and Y are *tangential externally* with "base sides" (fig. 14).

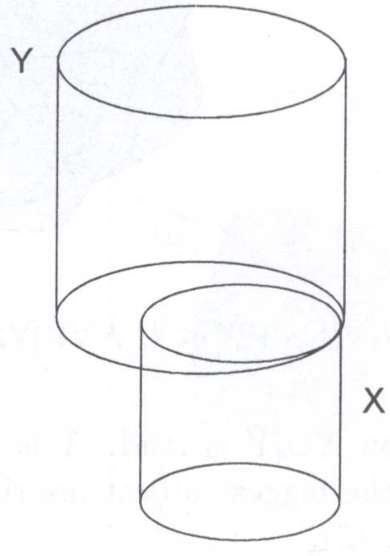

Fig. 14

Definition 24
$X\alpha_2 Y \Leftrightarrow X \leq Y \wedge \exists Z Z\alpha_1 X, Y.$

It is read: the cylinder X is a *part* of the cylinder Y and is *tangential* to it with the "base side" (fig. 15).

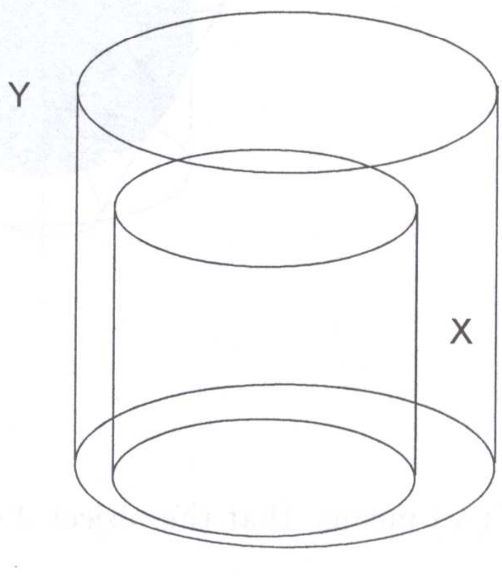

Fig. 15

Definition 25
$X\alpha_3YZ \Leftrightarrow \boldsymbol{W}(Z) \wedge X\alpha_1Y \wedge \forall U\alpha_1Y(X\alpha_2U \Rightarrow U\alpha_1Z)$

The expression $X\alpha_3YZ$ is read: the cylinder X is *tangent externally by the same "base side"* to cylinders Y and Z (fig. 16).

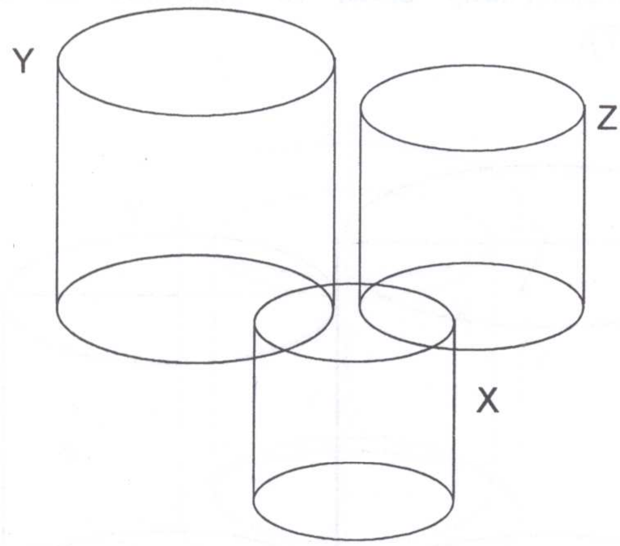

Fig. 16

Definition 26
$X\alpha_4Y \Leftrightarrow X = \sum_A[\exists U(A\omega_4U \wedge U\omega_2Y \wedge \exists VV\alpha_3UY)]$.

The layer X determined by the cylinder Y will be the mereogical sum of all the spheres which are inscribed into such cylinders which are congruent to the cylinder Y. For those cylinders a cylinder exist which is tangential externally by the "same base" side to those cylinders.

Definition 27
$\boldsymbol{V}(X) \Leftrightarrow \exists YX\alpha_4Y$

The expression $\boldsymbol{V}(X)$ is read: the object X is a *layer*.

Definition 28
$X\alpha_5Y \Leftrightarrow \exists Z(Z\alpha_3XY \vee \exists U[(U\alpha_1X \wedge Z\alpha_3UY) \vee (U\alpha_1Y \wedge Z\alpha_3UX)])$.

The expression $X\alpha_5Y$ is read: the cylinders X and Y have "parallel axes".

Definition 29
$x\alpha_6Y \Leftrightarrow x \leq Y \wedge \exists(u,v)(Y\varrho_{16}uv \wedge x \equiv u)$.

The above relation states: the square x is one of the squares which are generating the cylinder Y.

Definition 30
$X\alpha_7Y \Leftrightarrow \neg(X\clubsuit Y) \wedge \neg(X\alpha_1Y) \wedge X\alpha_4Y \wedge \exists(u,v)(u\alpha_5X \wedge v\alpha_5Y \wedge w\varrho_2v).$

This relation states about two cylinders which are tangent externally with "side surfaces". This situation is real when complanar squares exist generating both cylinders, which are tangent externally "sides" (we exclude cylinders tangent externally "sides" to "base side" or "base side" with "side surfaces") (fig. 17).

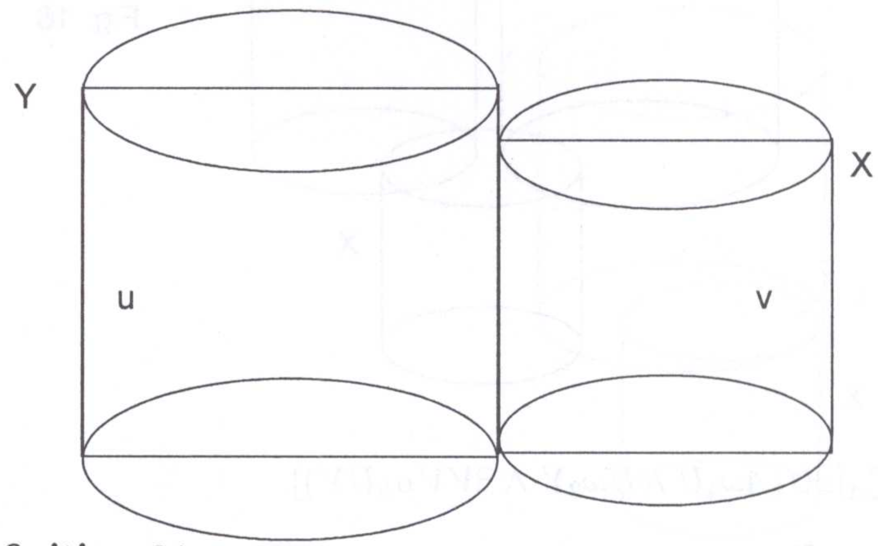

Fig. 17

Definition 31
$X\alpha_8Y \Leftrightarrow X\clubsuit Y \wedge \exists(u,v,w,Z)(u\alpha_6X \wedge v\alpha_6Y \wedge w\alpha_6Z \wedge Z\alpha_7X, Y \wedge w\varrho_4uv).$

In the relation α_8 cylinders X and Y are tangent internally "side surfaces" (fig. 18).

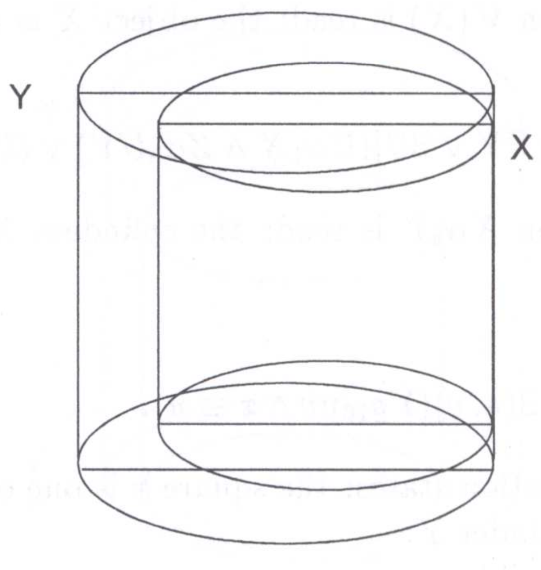

Fig. 18

Definition 32
$X\alpha_9 Y \Leftrightarrow W(X) \wedge W(Y) \wedge X \leq Y \wedge X \neq Y \wedge \exists U \forall V[V\omega_2 U \Rightarrow (V\alpha_7 X \Rightarrow V\alpha_8 U)]$.

The expression $X\alpha_9 Y$ is read: the cylinders X and Y have one axis and the cylinder X is a proper part of the cylinder Y (fig. 19).

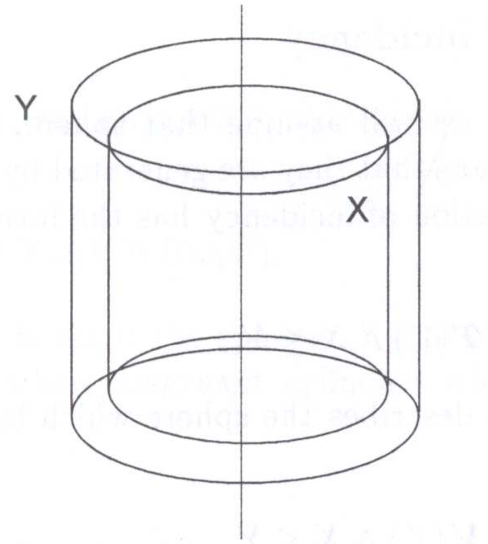

Fig. 19

Definition 33
$X\alpha_{10} Y \Leftrightarrow X\omega_2 Y \wedge \exists Z X, Y\alpha_9 Z$

Relation α_{10} tells about two common-tunnel cylinders X and Y (fig. 20).

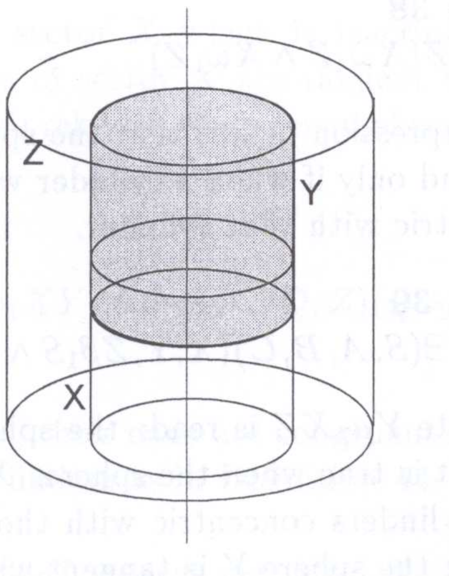

Fig. 20

Definition 34
$X\alpha_{11} Y \Leftrightarrow X = \sum_A [\exists Z(A\omega_4 Z \wedge Z\alpha_{10} Y)]$.

The expression $X\alpha_{11}Y$ is read: the X is a *tunnel determined by the cylinder* Y. It holds if and only if the X is the meorological sum of all spheres which are generated by the common-tunnel cylinders with the cylinder Y.

Definition 35
$\boldsymbol{T}(X) \Leftrightarrow \exists Y X\alpha_{11}Y$.

The symbol $\boldsymbol{T}(X)$ is read: the object X is a *tunnel.* The object X is a tunnel if and only if a cylinder Y which determined this tunnel exists.

4. Relation of incidency

In this paragraph we will assume that sphere, tunnel and layer have the same radius, it means that they are generated by congruent cylinders. With this condition, relation of incidency has the form:

Definition 36
$X\beta_1 Y \Leftrightarrow \boldsymbol{K}(X) \wedge \boldsymbol{T}(Y) \wedge X \leq Y$

This defnition describes the sphere which laying in the tunnel Y.

Definition 37
$X\beta_2 Y \Leftrightarrow \boldsymbol{K}(X) \wedge \boldsymbol{V}(Y) \wedge X \leq Y$

The expression $X\beta_2 Y$ is read: the sphere X lays at the layer Y.

5. Relation of in-between position

We will define relation of concentricity of the cylinder Y and the sphere X.

Definition 38
$X\mu_1 Y \Leftrightarrow \exists Z(X\omega_4 Z \wedge X\omega_1 Z)$.

This expression means that the sphere X and the cylinder Y are concentric if and only if when a cylinder which generates this sphere exist and it is concentric with that cylinder.

Definition 39
$Y\mu_2 XZ \Leftrightarrow \exists(S,A,B,C)(X,Y,Z\beta_1 S \wedge A\mu_1 X \wedge B\mu_1 Y \wedge C\mu_1 Z \wedge B\alpha_1 A, C)$.

The note $Y\mu_2 XZ$ is read: the sphere Y is *laying between* the spheres X and Z. It is true when the spheres X, Y, Z are laying in common-tunnel and such cylinders concentric with those spheres exist, that cylinder concentric with the sphere Y is tangent with external “base-side” to cylinders concentric with those spheres X and Z.

6. Relation of congruency of sectors

Now we introduce the notion of a closed-sector with two ends at spheres X and Y. It will be the set of all spheres laying between his ends with its ends too. The definition of a closed-sector has the form:

Definition 40
$Z\varphi_1 XY \Leftrightarrow X \neq Y \wedge \exists S X, Y\beta_1 S \wedge Z = \sum_A (A = X \vee A = Y \vee A\mu_2 XY)$.

Definition 41
$S(Z) \Leftrightarrow \exists (X, Y) Z\varphi_1 XY$

The note $S(Z)$ is read: the object Z is the *sector*.

Definition 42
$X\varphi_2 Y \Leftrightarrow \exists (U, V)(X\omega_4 V \wedge Y\omega_4 V \wedge U\omega_2 V)$.

The expression $X\varphi_2 Y$ is read: the spheres X and Y are *congruent*. It takes place if and only if when congruent cylinders which are generating those spheres exist.

Definition 43
$X\varphi_3 Y \Leftrightarrow \exists U(X\omega_4 U \wedge U\alpha_8 Y \wedge U\alpha_2 Y)$.

The expression $X\varphi_3 Y$ is read: the sphere X is a part of the cylinder Y and tangent with it at the center of one of the base sides.

Definition 44
$X\varphi_4 Y \Leftrightarrow \exists (A, B)(X\varphi_1 AB \wedge A, B\varphi_3 Y)$.

This relation characterises the sector X which is inscribed into the cylinder Y in such a way, that ends of sector X are tangent to center of cylinder's "bases". Now we can define relation congruency of sectors in this paper.

Definition 45
$A\varphi_5 B \Leftrightarrow \exists (U, V, X, Y)[A\varphi_1 UV \wedge B\varphi_1 XY \wedge U\varphi_2 X \wedge \exists (Q, Z)(Q\omega_2 Z \wedge A\varphi_4 Q \wedge B\varphi_4 Z)]$.

Two sectors are congruent when, their ends are congruent and we can inscribe them into two congruent cylinders as in Definition 44.

We have showed, that geometry of squares can give all the intuitive definitions equivalent to the elementary notions and the relations of the

Hilbert geometry. Let us remind, that the relations of congruent angles can be defined using congruency of sectors. System of axioms for the geometry of squares will be state partially of axioms in [10] and additionally Hilbert's axioms for Euclid geometry.

References

[1] K. Borsuk, W. Szmielew, *Podstawy geometrii*, PWN, Warszawa 1955.

[2] G. Bryll, R. Sochacki, *O dwóch różnych pojęciach zbioru*, Roczniki PTM - Dydaktyka Matematyki 19 (1997), 5–13.

[3] R. Ditchen, E. Glibowski, S. Kościk, *O pewnym układzie pojęć pierwotnych geomertii elementarnej*, Acta Universitatis Wratislavensis – Matematyka, Fizyka i Astronomia 4 (1963), 5–11.

[4] E. Glibowski, J. Słupecki, *Geometria sześcianów*, Zeszyty Naukowe WSP w Opolu - Matematyka 1 (1956), 38–47.

[5] E. Glibowski, *The application of mereology to grounding of elementary geometry*, Studia Logica, vol. XXIV (1969), 109–125.

[6] D. Hilbert, *Grundlagen der Geometrie*, Leipzig - Berlin 1930.

[7] S. Jaśkowski, *Geometria brył*, Matematyka 1 (3) (1949), 1–6.

[8] S. Leśniewski, *O podstawach matematyki*, Przegląd Filozoficzny t. 30-34 (1927-1931).

[9] J. Słupecki, *Towards a generalized mereology of Leśniewski*, Studia Logica vol. 8 (1958), 131–154.

[10] R. Sochacki, *Teoria kwadratów na gruncie mereologii Leśniewskiego*, Rozprawa doktorska. Uniwersytet Wrocławski, 1999.

[11] A. Tarski, *Foundations of the geometry of solids*, In: *Logic, Semantic, Metamethematics Papers from 1923 to 1938 by A. Tarski* (traslated by J.H. Woodger), Hackett Publishing Company 1983, 24–29.

Robert Sochacki, Leszek Jaworski
Uniwersytet Opolski
45-052 Opole
ul. Oleska 48