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SPECIAL PARTIAL ORDERINGS IN SIMPLE GRAPHS

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Abstract. We show an algorithm checking whether in a given simple graph G it is possible to introduce a partial ordering whose covering relation agrees with the adjacency relation in G.

We start with recalling some basic definitions of the graph theory.

An *(undirected) graph* $G = (V, E)$ consists of a non-empty set V, called the *vertex-set* and a set E of two element subset $\{u, v\}$ of the set V, called the *edge-set*.

Vertices u and v of a graph $G = (V, E)$ are *adjacent vertices* if $\{u, v\} \in E$. Then, it is said that they are joined by the edge $e = \{u, v\}$. We say that the vertices u and v are *incident* with the edge e, and the edge e is *incident* with the vertices u and v.

Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there exists a bijection $\phi : V \to V'$ such that for every $u, v \in V$

$$\{u, v\} \in E \iff \{\phi(u), \phi(v)\} \in E'.$$

A subgraph of a graph $G = (V, E)$ is a graph $G_1 = (V_1, E_1)$ such that $V_1 \subseteq V$ and $E_1 \subseteq E$.

Assume that $\mathcal{U} = \{A_i; 1 \leq i \leq n\}$ is a family of finite sets.
A sequence $(a_1, a_2, \ldots, a_n)$ such that $a_i \in A_i$ for $i = 1, 2, \ldots, n$ and $a_i \neq a_j$ for $i \neq j$ is called a transversal or a system of distinct representatives of the family $\mathcal{U}$.

Every transversal of a partition of the vertex-set V of the graph G determines a subgraph of G.

Now, we introduce the notion of an acyclic partition of the vertex-set V.

Definition 1. Let $G = (V, E)$ be a graph. The family $\mathcal{P}$ of non-empty, pairwise disjoint sets V_i, $i = 1, \ldots, n$, such that

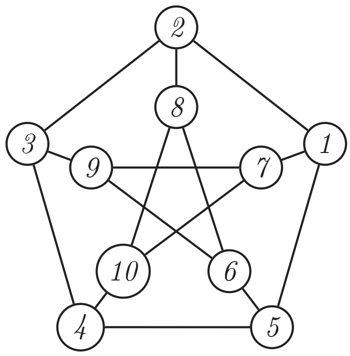
[1] $\bigcup_{i=1}^{n} V_i = V$;

[2] if $\{u, v\} \in E$, then $u \in V_i$ and $v \in V_j$ for $i \neq j; i, j \in \{1, \ldots, n\}$;

[3] none of transversals of the family $\mathcal{P}$ contains a cycle

is called an acyclic partition of the vertex-set V.

Example 1. *Consider the following graph G.*



$\mathcal{P}_1 = \{\{1, 3, 6, 10\}, \{2, 4, 7\}, \{5, 8, 9\}\}$;
$\mathcal{P}_2 = \{\{1, 6\}, \{2, 5\}, \{3, 7\}, \{4, 8\}, \{9, 10\}\}$
are acyclic partitions of the vertex-set of G.

Theorem 1. *A graph $G = (V, E)$, in which there exists a cycle of length 3, does not possess an acyclic partition of the vertex-set V.*

Proof. If $\mathcal{P} = \{V_i\}_{1 \leq i \leq n}$ is any acyclic partition of V then, according to the definition, v_1, v_2 and v_3 have to belong to different elements, let say V_1, V_2, V_3 of the partition. However, choosing v_1, v_2 and v_3 as the representatives of V_1, V_2 and V_3 we violate the third condition of the definition. □

By an ordered partition of the set V we mean a sequence $(V_i)_{1 \leq i \leq n}$ of disjoint subsets of V such that $\bigcup_{i=1}^{n} V_i = V$.
Let $G = (V, E)$ be a graph, for which there exists an acyclic ordered partition $\mathcal{P} = (V_i)_{1 \leq i \leq n}$ of the vertex-set V. Let $f_{\mathcal{P}}(v) = i$ for $v \in V_i$.

Theorem 2. *Let $G = (V, E)$ be a graph, for which there exists an acyclic ordered partition $\mathcal{P} = (V_i)_{1 \leq i \leq n}$ of the vertex-set V. Then a binary relation $<_{\mathcal{P}}$ on V defined by*

$u <_{\mathcal{P}} v$ iff there exists a path d between u and v such that $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(v)$ and for every $w \in V \setminus \{u, v\}$ if $w \in d$, then $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(w) <_{\mathcal{P}} f_{\mathcal{P}}(v)$

is a strong partial ordering on V.

Proof. It is clear that the relation $<_{\mathcal{P}}$ is antireflexive and antisymmetric. We will show that it is transitive. Let $u, v, w \in V$ and $u <_{\mathcal{P}} v, v <_{\mathcal{P}} w$. Then there exist the paths d_1 and d_2 between u and v and between v and w respectively, such that $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(v)$ and for every $a \in V \setminus \{u, v\}$ if $a \in d_1$ then $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{<_{\mathcal{P}}}(a) <_{\mathcal{P}} f_{\mathcal{P}}(v)$ and $f_{\mathcal{P}}(v) <_{\mathcal{P}} f_{\mathcal{P}}(w)$ and for every $b \in V \setminus \{v, w\}$ if $b \in d_2$ then $f_{\mathcal{P}}(v) <_{\mathcal{P}} f_{\mathcal{P}}(b) <_{\mathcal{P}} f_{\mathcal{P}}(w)$. Thus, there exists the path d between u and w such that $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(w)$ and for every $c \in V \setminus \{u, w\}$ if $c \in d$, then $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(c) <_{\mathcal{P}} f_{\mathcal{P}}(w)$. We obtain $u <_{\mathcal{P}} w$. Therefore the relation $<_{\mathcal{P}}$ is transitive. □

Let $G = (V, E)$ be a graph with an acyclic ordered partition $\mathcal{P}$ and let the relation $\prec_{\mathcal{P}}$ be the covering relation associated to the relation $<_{\mathcal{P}}$ from Theorem 2.

Theorem 3. *Let $G = (V, E)$ be a graph with the acyclic ordered partition $\mathcal{P}$. For every $u, v \in V$ we have*

$$u \prec_{\mathcal{P}} v \Leftrightarrow (\{u, v\} \in E \wedge f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(v)).$$

Proof. Let's assume that v covers u. Thus $u <_{\mathcal{P}} v$ and there is no vertex w of V with $u <_{\mathcal{P}} w <_{\mathcal{P}} v$. Then there exists a path d between u and v and $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(v)$. Since none of paths between u and v goes through the vertex w different from u and v, thus the path d is of length 1. Therefore $\{u, v\} \in E$ and $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(v)$.

Now, we assume that $\{u, v\} \in E$ and $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(v)$. Since none of transversals of the partition $\mathcal{P}$ contains a cycle, thus the edge $\{u, v\}$ is the only one path between u and v. Therefore v covers u. □

Corollary 1. *If in a graph G there is a cycle of length 3, then it is not possible to introduce a partial ordering whose covering relation agrees with the adjacency relation in G.*

We show an algorithm for finding an acyclic partition of the vertex-set V in the graph $G = (V, E)$, $|V| = n$.

[1] Let V be a vertex-set in graph $G = (V, E)$ and let $(V_i)_{1 \leq i \leq n}$ be a sequence of sets. We assume that $V_i = \emptyset$ for every $1 \leq i \leq n$.

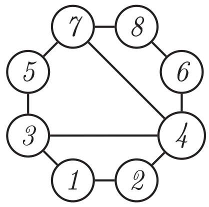
[2] While $V \neq \emptyset$ do:
choose any vertex $v \in V$. Look for i such that none of vertices of the set V_i is joined by an edge with the vertex v and none of transversals of the family of sets $V_1, \ldots, V_{i-1}, V_i \cup \{v\}, V_{i+1}, \ldots, V_n$ contains a cycle.

- if there exists such a set V_i, then we remove the vertex v from the set V and add it into the set V_i.
- else there is not an acyclic partition of the vertex-set V.

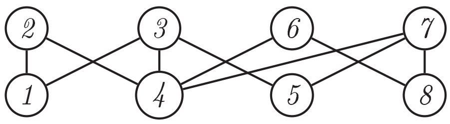
[3] The family $\mathcal{P}$ of non-empty, pairwise disjoint sets V_i of the sequence $(V_i)_{1 \leq i \leq n}$ is an acyclic partition of the vertex-set V.

It is clear that using this algorithm we can obtain many different acyclic partitions of the vertex-set V, depending on the order of chosen elements.

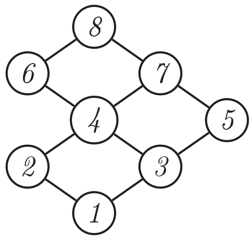
Example 2. *We use this algorithm to introduce a partial ordering relation in the following graph.*



Using this algorithm, we can obtain the following acyclic partition of the vertex-set V: $V_1 = \{1,4,5,8\}$, $V_2 = \{2,3,6,7\}$. In this case, we have the following Hasse diagram



We can also obtain the following acyclic partition of the vertex-set V : $V_1 = \{1\}$, $V_2 = \{2,3\}$, $V_3 = \{4,5\}$, $V_4 = \{6,7\}$, $V_5 = \{8\}$ and now, the Hasse diagram looks as follow



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