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Certain Classes of Elimination Operators

Grzegorz Bryll, Robert Sochacki

Let S be a non-empty set. Operator $E : 2^S \to 2^S$ is called an *elimination operator*, when for any $X, Y \in 2^S$ the following conditions are fulfilled (compare [3], [4]):

$$\begin{aligned} E(X) &\subseteq X, \\ X \subseteq Y &\Rightarrow E(X) \subseteq E(Y), \\ E(X) &\subseteq E(E(X)). \end{aligned} \tag{1}$$

The result of the formulas is that $E(E(X)) = E(X)$.

Operator E is general weaker than operator Int of topological interior, which is characterised by the following conditions (compare [2]):

$$\begin{aligned} Int(X) &\subseteq X, \\ Int(X \cap Y) &= Int(X) \cap Int(Y), \\ Int(X) &\subseteq Int(Int(X)), \\ Int(S) &= S, \\ \text{for every } X, Y &\in 2^S \end{aligned} \tag{2}$$

In this paper, the class of all elimination operators over a given universum shall be researched as well as the class of completely multiplication elimination operators and isomorphic relationships of these classes with certain classes of sets families shall be analysed. Elimination operators are useful and applicable to researches concerning refutation of formulas [1].

Class of all elimination operators over a given universum

Family of theories (systems) of any elimination operator E is defined in the following way:

Definition 1
$Th(E) = \{X \in 2^S : E(X) = X\}$.

Thus, theory is any set X, closed due to E operator. Theories are for example sets $\emptyset$ and $E(X)$.

The set of all elimination operators is marked by Ω, i.e.:

Definition 2
$\Omega = \{E | E : 2^S \to 2^S$ *and* E *is an elimination operator in* $S\}$.

It easily can be proved that:

Lemma 1
$(E_1, E_2 \in \Omega \wedge E_1 \neq E_2) \Rightarrow Th(E_1) \neq Th(E_2)$.

In the set Ω, an order relation can be introduced by adopting the following definition:

Definition 3
$E_1 \leq E_2 \Leftrightarrow \forall X \subseteq S\, (E_1(X) \subseteq E_2(X)),\ \ E_1, E_2 \in \Omega$.

It is visible that the relation $\leq$ is reflexive, anti-symmetric and transitive, thus it constitutes an order.

Between elimination operators and their relative families of theories, the following relation appears:

Lemma 2
$E_1 \leq E_2 \Leftrightarrow Th(E_1) \subseteq Th(E_2)$, *for every* $E_1, E_2 \in \Omega$.

This lemma clearly results from assumed definitions and formula (1).

Let us notice that on the basis of any family $\mathrm{R} \subseteq 2^S$, a relative elimination operator E_{R} can be constructed by adopting the following definition:

Definition 4
$E_R(X) = \bigcup\{Y : Y \subseteq X \wedge Y \in R\}$, *for any* $X \in 2^S$

E_{R} operator fulfils the conditions given in formula (1). It shall be called elimination operator based on R. Of course, $E_{\mathrm{R}} \in \Omega$ if $\mathrm{R} \subseteq 2^S$.

Let us consider the following class of families of sets:

Definition 5
$\mathcal{K} = \{R \subseteq 2^S : \forall R_1 (R_1 \subseteq R \Rightarrow \bigcup R_1 \in R)\}$.

Thus, class $\mathcal{K}$ is the class of families closed for summation operation.

It shall be proved that:

Lemma 3
$R \in \mathcal{K} \Rightarrow Th(E_R) = R.$

Proof:

Let us assume that $R \in \mathcal{K}$ and $X \in Th(E_R)$. Thus, due to Definitions 1 and 4, we obtain: $X = E_R(X) = \bigcup\{Y : Y \subseteq X \wedge Y \in R\}$. The family $\{Y : Y \subseteq X \wedge Y \in R\}$ is marked by R_1. Since $R_1 \subseteq R$, thus, due to the assumption $R \in \mathcal{K}$ and Definition 5, we receive $\bigcup R_1 \subseteq R$, namely $X \in R$. So, we have demonstrated inclusion $Th(E_R) \subseteq R$.

Let us then assume that $Z \in R$. Because $Z \subseteq \bigcup\{Y : Y \subseteq Z \wedge Y \in R\}$, thus due to Definition 4: $Z \subseteq E_R(Z)$. However, on the basis of formula (1) we obtain $E_R(Z) \subseteq Z$ and simultaneously $E_R(Z) = Z$, namely $Z \in Th(E_R)$. Thus, inclusion $R \subseteq Th(E_R)$ was proved, it completes the proof.

Lemma 4
$E \in \Omega \Rightarrow Th(E) \in \mathcal{K}.$

For any elimination operator, the family of theories of that operator is thus closed because of summation. Thus, the sum of any family of theories (for a given E) is also a theory[1].

The following theorem giving a relationship between ordered sets Ω and $\mathcal{K}$:

Theorem 1
Function $\varphi : \Omega \to \mathcal{K}$ defined by the formula

$$\varphi(E) = Th(E) \qquad (3)$$

is an isomorphism between relational systems $(\Omega, \leq)$ and $(\mathcal{K}, \subseteq)$.

Proof:

The result from Lemma 1 and formula (3) is that $Th(E) \in \mathcal{K}$, for every $E \in \Omega$. On the basis of Lemma 1, it can be claimed that function φ is injection, but on the basis of Lemma 3, the function is surjection. Moreover, taking into consideration Lemma 2, we come to the conclusion that φ is isomorphism, it completes the proof.

[1]Dual thesis proceeds consequence operator: multiplication of any family of theories (due to consequence operator C) is a theory, too.

Class of completely multiplicatory elimination operators

Let us assume two following definitions:

Definition 6
$\mathcal{K}_{\cap} = \{R \in \mathcal{K} : \forall R_1 (R_1 \subseteq R \Rightarrow \bigcap R_1 \in R)\}$.

Thus, the class $\mathcal{K}_{\cap}$ is of all closed families due to summation and simultaneously closed due to multiplication operation.

Definition 7
$\Omega_M = \{E \in \Omega : \forall T \subseteq 2^S\ [\bigcap\{E(X) : X \in T\} \subseteq E(\bigcap T)]\}$.

Class Ω_M shall be called the class of all completely multiplicatory elimination operators.

Further, the following lemma shall be used:

Lemma 5
For any $E \in \Omega$ *and any* $T \subseteq 2^S$*:*

$E(\bigcap T) \subseteq \bigcap\{E(X) : X \in T\}$.

It shall be proved that:

Lemma 6
$E \in \Omega_M \Rightarrow Th(E) \in \mathcal{K}_{\cap}$.

Proof:

From the assumption $E \in \Omega_M$ and Definitions 7, 8, the following formulas result: $E \in \Omega$ and $\bigcap\{E(X) : X \in \mathrm{T}\} \subseteq E(\bigcap \mathrm{T})$, for every $\mathrm{T} \subseteq 2^S$.

Let us assume, additionally, that $\mathrm{R}_1 \subseteq Th(E)$. Since $\mathrm{R}_1 \subseteq 2^S$, thus $\bigcap\{E(X) : X \in \mathrm{R}_1\} \subseteq E(\bigcap \mathrm{R}_1)$. If, however, $X \in \mathrm{R}_1$ is, on the basis of Definition 1 and additional assumption, we obtain $E(X) = X$, and simultaneously $\bigcap\{X : X \in \mathrm{R}_1\} \subseteq E(\bigcap \mathrm{R}_1)$. On the basis of Lemma 5 and additional assumption, we also obtain inclusion $E(\bigcap \mathrm{R}_1) \subseteq \bigcap\{X : X \in \mathrm{R}_1\}$. Two last inclusions give the following equation $E(\bigcap \mathrm{R}_1) = \bigcap \mathrm{R}_1$, which gives the result $\bigcap \mathrm{R}_1 \in Th(E)$. By this, implication $\mathrm{R}_1 \subseteq Th(E) \Rightarrow \bigcap \mathrm{R}_1 \in Th(E)$ was proved for any $\mathrm{R}_1 \subseteq 2^S$. On the basis of Definition 6, it is claimed that $Th(E) \in \mathcal{K}_{\cap}$, it completes the proof.

Lemma 7
$R \in \mathcal{K}_{\cap} \Rightarrow E_R \in \Omega_M$.

Elimination operator based on R, which belongs to $\mathcal{K}_{\cap}$, constitutes thus completely multiplicatory operator.

Proof:

It results from the assumption $\mathrm{R} \in \mathcal{K}_\cap$ that $\mathrm{R} \in \mathcal{K}$, where from Lemma 3 we receive $Th(E_\mathrm{R}) = \mathrm{R}$. Because $E_\mathrm{R}(X) = \bigcup\{Y : Y \subseteq X \wedge Y \in \mathrm{R}\}$ (Definition 4), thus $E_\mathrm{R}(\bigcap \mathrm{T}) = \bigcup\{Y : Y \subseteq \bigcap \mathrm{T} \wedge Y \in Th(E_\mathrm{R})\} = \bigcup\{Y : Y \subseteq \bigcap \mathrm{T} \wedge E_\mathrm{R}(Y) = Y\}$, for any $\mathrm{T} \subseteq 2^S$.

Let $\mathrm{R}_1 = \{E_\mathrm{R}(X) : X \in \mathrm{T}\}$ and let us additionally assume that $Z \in \mathrm{R}_1$. Thus $Z = E_\mathrm{R}(X_1)$ for some $X_1 \in \mathrm{T}$. Thus, we obtain $E_\mathrm{R}(Z) = E_\mathrm{R}(E_\mathrm{R}(X_1)) = E_\mathrm{R}(X_1) = Z$, namely, $Z \in Th(E_\mathrm{R})$, and hence $Z \in \mathrm{R}$. Thus, inclusion $\mathrm{R}_1 \subseteq \mathrm{R}$ was justified, which, on the basis of initial assumption and Definition 6, $\bigcap \mathrm{R}_1 \in \mathrm{R}$ is obtained. Thus, due to formula $Th(E_\mathrm{R}) = \mathrm{R}$, we obtain that $\bigcap \mathrm{R}_1 = E_\mathrm{R}(\bigcap \mathrm{R}_1)$. Taking into consideration the assumption, the last equation may be written as: $\bigcap\{E_\mathrm{R}(X) : X \in \mathrm{T}\} = E_\mathrm{R}(\bigcap\{E_\mathrm{R}(X) : X \in \mathrm{T}\})$. Since $E_\mathrm{R}(X) \subseteq X$ for every $X \subseteq S$, thus $\bigcap\{E_\mathrm{R}(X) : X \in \mathrm{T}\} \subseteq \bigcap\{X : X \in \mathrm{T}\} = \bigcap \mathrm{T})$, from which the following results: $E_\mathrm{R}(\bigcap\{E_\mathrm{R}(X) : X \in \mathrm{T}\} \subseteq E_\mathrm{R}(\bigcap \mathrm{T})$. Thus it was proved that $\bigcap\{E_\mathrm{R}(X) : X \in \mathrm{T}\} \subseteq E_\mathrm{R}(\bigcap \mathrm{T})$, for any $\mathrm{T} \subseteq 2^S$. Hence, on the basis of Definition 7, it is claimed that $E_\mathrm{R} \in \Omega_M$.

The relationship between Ω_M and $\mathcal{K}_\cap$ classes provides the following theorem.

Theorem 2
Function $\psi : \Omega_M \to \mathcal{K}_\cap$ defined by the formula

$$\psi(E) = Th(E) \tag{4}$$

is an isomorphism between relational systems $(\Omega_M, \leq)$ and $(\mathcal{K}_\cap, \subseteq)$.

Proof:

On the basis of Lemma 6 and formula (4), we obtain: $E \in \Omega_M \Rightarrow \psi(E) \in \mathcal{K}_\cap$. It results from Lemma 1 that function ψ is injection, however, on the basis of Lemma 7 we claim that this function is surjection. Moreover, the following equation proceeds: $E_1 \leq E_2 \Leftrightarrow \psi(E_1) \subseteq \psi(E_2)$, for any $E_1, E_2 \in \Omega_M$ (see Lemma 2). Thus function ψ is an isomorphism between $(\Omega_M, \leq)$ and $(\mathcal{K}_\cap, \subseteq)$.

The result from the Theorems 2 and 3 is the following corollary.

Corollary 1 $\psi = \varphi/\Omega_M$.

References

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Opole University
Institute of Mathematics
Oleska 48,
45-052 Opole