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Varieties of Ordered Semigroups

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1. Introduction

It is known that there is a bijection between all varieties of semigroups and all fully invariant congruences on the free semigroup on the set of variables $\{x_1, x_2, x_3, \ldots\}$. It was done (more generally, for varieties of algebras) by Birkhoff in 1930's. The reader can consult Birkhoff's book [1]. So, to any variety of semigroups we have its syntactical counterpart.

In this contribution we present syntactical counterparts to varieties of ordered semigroups. Our main theorem (2.6) asserts that there is a bijection between all varieties of ordered semigroups and all fully invariant stable quasiorders on the free semigroup on the set $\{x_1, x_2, x_3, \ldots\}$ (fully invariant stable quasiorders are, roughly speaking, fully invariant congruences which do not need to be symmetric). We use the notion of identities for ordered algebras introduced by Bloom in [2]. Bloom has proved that varieties of ordered algebras are exactly classes defined by identities ([2], Theorem 2.6.). The special case of this result (when the ordered algebras are ordered semigroups) is our corollary 2.7

A structure $(S, \cdot, \leq)$ is called an ordered semigroup if

(i) $(S, \cdot)$ is a semigroup

(ii) $(S, \leq)$ is a (partially) ordered set

(iii) for any $a, b, c \in S$, $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$.

Let $S = (S, \cdot_S, \leq_S)$ and $T = (T, \cdot_T, \leq_T)$ be ordered semigroups. By a homomorphism $h : (S, \cdot_S, \leq_S) \to (T, \cdot_T, \leq_T)$ we mean a mapping $h : S \to T$ such that, for any $a, b \in S$, $h(a \cdot_S b) = h(a) \cdot_T h(b)$ and $a \leq_S b$ implies $h(a) \leq_T h(b)$. As usual, the ordered semigroup T is said to be a homomorphic image of S if there exists a surjective homomorphism $h : S \to T$. Further, T is said to be a substructure of S if $T \subseteq S$ and, for any $a, b \in T$, $a \cdot_T b = a \cdot_S b$, $a \leq_T b$ if and only if $a \leq_S b$.

Let $\{(S_i, \cdot_i, \leq_i)|\ i \in I\}$ be a set of ordered semigroups. By a product $\prod_{i\in I}(S_i, \cdot_i, \leq_i)$ we mean an ordered semigroup $(\prod_{i\in I} S_i, \cdot, \leq)$ where $(s_i)_{i\in I} \cdot (t_i)_{i\in I} = (s_i \cdot_i t_i)_{i\in I}$ and $(s_i)_{i\in I} \leq (t_i)_{i\in I}$ if and only if $s_i \leq_i t_i$ for all $i \in I$.

Let $\mathcal{V}$ be a class of ordered semigroups. Denote by
$\mathrm{H}(\mathcal{V})$ — the class of all homomorphic images of members of $\mathcal{V}$
$\mathrm{S}(\mathcal{V})$ — the class of all substructures of members of $\mathcal{V}$
$\mathrm{P}(\mathcal{V})$ — the class of all products of members of $\mathcal{V}$.
We say that $\mathcal{V}$ is a variety provided $\mathrm{H}(\mathcal{V}) \subseteq \mathcal{V}$, $\mathrm{S}(\mathcal{V}) \subseteq \mathcal{V}$ and $\mathrm{P}(\mathcal{V}) \subseteq \mathcal{V}$.

Let us denote the variety of all ordered semigroups by **OS**.

2. Identities for ordered semigroups

Let Y be a non-empty set. The free semigroup on Y will be denoted by Y^+.

An identity is any ordered pair $u \preceq v$ of words $u, v \in Y^+$. Let $(S, \cdot, \leq)$ be an ordered semigroup. We say that the identity $u \preceq v$ is satisfied in $(S, \cdot, \leq)$ if, for any homomorphism $\varphi : Y^+ \to (S, \cdot)$, $\varphi(u) \leq \varphi(v)$. An identity is satisfied in a class $\mathcal{V}$ of ordered semigroups if it is satisfied in any member of $\mathcal{V}$.

Let $\mathcal{V} \subseteq \mathbf{OS}$ be a class of ordered semigroups. We put

$$\varrho(\mathcal{V}, Y) = \{(u, v) \in Y^+ \times Y^+ |\ \text{the identity } u \preceq v \text{ is satisfied in } \mathcal{V}\}.$$

Let $\Sigma \subseteq Y^+ \times Y^+$ be a set of identities. We put

$$[\Sigma] = \{(S, \cdot, \leq) \in \mathbf{OS}|\ (S, \cdot, \leq) \text{ satisfies all identities from } \Sigma\}.$$

Let $S = (S, \cdot)$ be a semigroup. A binary relation ρ on S is said to be

- a quasiorder if it is a reflexive and transitive relation
- stable if, for any $a, b, c \in S$, $a\rho b$ implies $ca\rho cb$ and $ac\rho bc$
- fully invariant if, for any endomorphism $\varphi : S \to S$ and any $a, b \in S$, $a\rho b$ implies $\varphi(a)\rho\varphi(b)$.

The set of all fully invariant stable quasiorders on S will be denoted by $\mathrm{FISQ}(S)$.

2.1 Lemma. *Let $\Sigma \subseteq Y^+ \times Y^+$. Then $[\Sigma]$ is a variety of ordered semigroups.*

PROOF. It is easy to show the inclusions
$\mathrm{H}([\Sigma]) \subseteq [\Sigma], \mathrm{S}([\Sigma]) \subseteq [\Sigma], \mathrm{P}([\Sigma]) \subseteq [\Sigma]$.

2.2 Lemma. *Let $\mathcal{V} \subseteq$ **OS**. Then $\varrho(\mathcal{V}, Y) \in \mathrm{FISQ}(Y^+)$.*

PROOF. It is easy.

Let ρ be a stable quasiorder on a semigroup $S = (S, \cdot)$. We construct an ordered semigroup $S/\rho = (S/\sim_\rho, \cdot, \leq)$. We define a relation $\sim_\rho$ on S in this way:

$$a \sim_\rho b \Longleftrightarrow a\rho b, b\rho a \ (a, b \in S).$$

It is easy to show that the relation $\sim_\rho$ is a congruence on $(S, \cdot)$. The congruence $\sim_\rho$ determines the semigroup $(S/\sim_\rho, \cdot)$. We define a relation $\leq$ on $S/\sim_\rho$ in the following way:

$$(a \sim_\rho) \leq (b \sim_\rho) \Longleftrightarrow a\rho b \ (a, b \in S).$$

We easily check that the relation $\leq$ on $S/\sim_\rho$ is correctly defined. Further, $(S/\sim_\rho, \cdot, \leq)$ is an ordered semigroup. We will denote it by S/ρ.

Let $\mathcal{V}$ be a class of ordered semigroups. By a free object in $\mathcal{V}$ on a non-empty set Z we mean a pair (S, ι), where $S \in \mathcal{V}$ and $\iota : Z \to S$ is a mapping with the following universal property: for any ordered semigroup $T \in \mathcal{V}$ and any mapping $\vartheta : Z \to T$ there exists a unique homomorphism $\psi : (S, \cdot, \leq) \to (T, \cdot, \leq)$ such that $\psi \circ \iota = \vartheta$. In the cases when the mapping ι is obvious we will omit it and we will simply say that S is a free object in $\mathcal{V}$ on Z. Notice that in any class of ordered semigroups there is, up to isomorphism, at most one free object on a given non-empty set.

2.3 Theorem. *Let $\mathcal{V}$ be a variety of ordered semigroups. Then $Y^+/\varrho(\mathcal{V}, Y)$ is a free object in $[\varrho(\mathcal{V}, Y)]$ on Y and $Y^+/\varrho(\mathcal{V}, Y) \in \mathcal{V}$. In particular, $Y^+/\varrho(\mathcal{V}, Y)$ is a free object in $\mathcal{V}$ on Y.*

PROOF. In the case $\varrho(\mathcal{V}, Y) = Y^+ \times Y^+$ the assertion of the theorem clearly holds. So, let $\varrho(\mathcal{V}, Y) \neq Y^+ \times Y^+$. Let $\{(u_i, v_i) \in Y^+ \times Y^+ | \ i \in I\}$ be the set of all identities over Y which are not satisfied in $\mathcal{V}$. For any $i \in I$, let us choose $(S_i, \cdot, \leq) \in \mathcal{V}$ and a homomorphism $\varphi_i : Y^+ \to (S_i, \cdot)$ so that $\varphi_i(u_i) \not\leq \varphi_i(v_i)$. Put $(S, \cdot, \leq) = \prod_{i \in I}(S_i, \cdot, \leq)$. Clearly, $(S, \cdot, \leq) \in \mathcal{V}$. Further, let us consider a homomorphism $\varphi = (\varphi_i)_{i \in I} : Y^+ \to (S, \cdot)$. It holds for $u, v \in Y^+$: $\varphi(u) \leq \varphi(v) \Leftrightarrow u\varrho(\mathcal{V}, Y)v$. Put $T = \varphi(Y^+)$. Then T is a subsemigroup in $(S, \cdot)$. We order the semigroup $(T, \cdot)$ by the restriction of the ordering from $(S, \cdot, \leq)$. We have $(T, \cdot, \leq) \in \mathcal{V}$ and $(T, \cdot, \leq) \cong Y^+/\varrho(\mathcal{V}, Y)$. Thus $Y^+/\varrho(\mathcal{V}, Y) \in \mathcal{V}$. It remains to prove that $Y^+/\varrho(\mathcal{V}, Y)$ is a free object in $[\varrho(\mathcal{V}, Y)]$ on Y. Let $(P, \cdot, \leq) \in [\varrho(\mathcal{V}, Y)], \vartheta : Y \to P$. Let us assume that $\psi : Y^+/\varrho(\mathcal{V}, Y) \to (P, \cdot, \leq)$ is a homomorphism satisfying $\psi \circ \iota = \vartheta$. At the same time the mapping $\iota : Y \to Y^+/\sim_{\varrho(\mathcal{V},Y)}$ is given by the rule $\iota(y) = y \sim_{\varrho(\mathcal{V},Y)} (y \in Y)$. For any $u \in Y^+$, $\psi(u \sim_{\varrho(\mathcal{V},Y)}) = \theta(u)$ where $\theta : Y^+ \to (P, \cdot)$ is the homomorphism extending the mapping

ϑ. It can be easily shown that the rule $\psi(u \sim_{\varrho(\mathcal{V},Y)}) = \theta(u)$ determines correctly a homomorphism $\psi : Y^+/\varrho(\mathcal{V},Y) \to (P,\cdot,\leq)$.

2.4 Lemma. *Let $\rho \in \mathrm{FISQ}(Y^+)$. Then $\rho = \varrho([\rho], Y)$.*

PROOF. Clearly, $\rho \subseteq \varrho([\rho], Y)$. So, we will prove the inclusion $\varrho([\rho], Y) \subseteq \rho$.We will show that $Y^+/\rho \in [\rho]$. Let $u, v \in Y^+, u\rho v, \varphi : Y^+ \to Y^+/\sim_\rho$ be a homomorphism. We want to show that $\varphi(u) \leq \varphi(v)$. For any $y \in Y$ let us choose $\vartheta(y) \in Y^+$ so that $\varphi(y) = \vartheta(y) \sim_\rho$. Let $\theta : Y^+ \to Y^+$ be the endomorphism extending the mapping $\vartheta : Y \to Y^+$. Let $y_{i_1}, \ldots, y_{i_k} \in Y$. Then

$$\begin{aligned}\varphi(y_{i_1} \ldots y_{i_k}) &= \varphi(y_{i_1}) \ldots \varphi(y_{i_k}) \\ &= (\vartheta(y_{i_1}) \sim_\rho) \ldots (\vartheta(y_{i_k}) \sim_\rho) \\ &= (\vartheta(y_{i_1}) \ldots \vartheta(y_{i_k})) \sim_\rho \\ &= \theta(y_{i_1} \ldots y_{i_k}) \sim_\rho .\end{aligned}$$

We have shown that, for any $w \in Y^+$, $\varphi(w) = \theta(w) \sim_\rho$. So, we want to prove that $(\theta(u) \sim_\rho) \leq (\theta(v) \sim_\rho)$, i.e. $\theta(u)\rho\theta(v)$. But it holds since $\rho \in \mathrm{FISQ}(Y^+)$. Now, let $(u, v) \in \varrho([\rho], Y)$. We want to show that $u\rho v$. The identity $u \preceq v$ is satisfied in the ordered semigroup Y^+/ρ. Let us consider the following homomorphism $\varphi : Y^+ \to Y^+/\sim_\rho, \varphi(w) = w \sim_\rho \ (w \in Y^+)$. Then $\varphi(u) \leq \varphi(v), (u \sim_\rho) \leq (v \sim_\rho), u\rho v$.

2.5 Lemma. *Let Y be an infinite set, $\mathcal{V}$ be a variety of ordered semigroups. Then $\mathcal{V} = [\varrho(\mathcal{V}, Y)]$.*

PROOF. Clearly, $\mathcal{V} \subseteq [\varrho(\mathcal{V}, Y)]$. So, we will prove the inclusion $[\varrho(\mathcal{V}, Y)] \subseteq \mathcal{V}$. Let $(S,\cdot,\leq) \in [\varrho(\mathcal{V}, Y)]$. Let us consider the set $Z = Y \cup S$. By 2.3, $Z^+/\varrho(\mathcal{V}, Z)$ is a free object in $[\varrho(\mathcal{V}, Z)]$ on Z and $Z^+/\varrho(\mathcal{V}, Z) \in \mathcal{V}$. Let $\vartheta : Z \to S$ be a surjective mapping. Since Y and Z are infinite sets, $[\varrho(\mathcal{V}, Y)] = [\varrho(\mathcal{V}, Z)]$, and so $(S,\cdot,\leq) \in [\varrho(\mathcal{V}, Z)]$. Let $\varphi : Z^+/\varrho(\mathcal{V}, Z) \to (S,\cdot,\leq)$ be the homomorphism satisfying $\varphi \circ \iota = \vartheta$ $(\iota : Z \to Z^+/\sim_{\varrho(\mathcal{V},Z)}, \iota(z) = z \sim_{\varrho(\mathcal{V},Z)})$. Necessarily, φ is a surjection. Thus $(S,\cdot,\leq) \in \mathcal{V}$.

2.6 Theorem. *Let Y be an infinite set. The rules*

$$\rho \mapsto [\rho], \ \mathcal{V} \mapsto \varrho(\mathcal{V}, Y)$$

determine mutually inverse order reversing bijections between all varieties of ordered semigroups and all fully invariant stable quasiorders on Y^+.

PROOF. The theorem follows immediately from 2.4 and 2.5.

2.7 Corollary. *Let $\mathcal{V}$ be a class of ordered semigroups. Then $\mathcal{V}$ is a variety of ordered semigroups if and only if there exist a non-empty set Y and a set of identities $\Sigma \subseteq Y^+ \times Y^+$ such that $\mathcal{V} = [\Sigma]$.*

The theorem 2.6 is an analogy for ordered semigroups with the classical result by Birkhoff (see, e.g., Theorem 22 in [1]) concerning the relationship between varieties of algebras of a given type and fully invariant congruences on a free algebra of words of the same type. The corollary 2.7 follows also from a result by Bloom presented in his paper on varieties of ordered algebras ([2], Theorem 2.6.).

References

[1] G.Birkhoff, *Lattice Theory*, Providence, Rhode Island, 1967

[2] S.L.Bloom, *Varieties of ordered algebras*, Journal of Computer and System Sciences 13, 200 – 212 (1976)

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