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ON PARA – ASSOCIATIVE BCI – ALGEBRAS

BCK – algebras were introduced as an algebraic formulation of C.A. Meredith's BCK – implicational calculus by Professor Kiyoshi Iséki in 1966 [6]. They form a quasivariety of algebras amongst whose subclasses can be found the earlier implicational models of Henkin [3], algebras of sets closed under set – subtraction, and dual relatively pseudocomplemented upper semilattices. Many of the articles in the Mathematics Seminar Notes of Kobe University, volume 3(1975) onwards, are devoted to these algebras. Iséki and Tanaka have provided many examples and the fundamental first – order theory in [9], [10], [11], while Iséki's survey contains many references [7].

The notion of BCI – algebras is a generalization of BCK – algebras [6]. Any group in which the square of every element is the indentity (i. e. a Boolean group) is a BCI – algebra but it is not a BCK – algebra. A BCI – algebra is a Boolean group if and only if it satisfies the identity $x*(y*x)=y$ (see[1]). In [5] is proved that a BCI – algebra $(G,*,0)$ of type $(2,0)$ is a Boolean group if it is associative.

In this note, we generalize this result. Namely, we prove that every para – associative BCI – algebra is a Boolean group.

By a *para – associative groupoid* we mean a groupoid $(G,*)$ with the following identity:

$$(x_1*x_2)*x_3=x_i*(x_j*x_k), \tag{1}$$

where $\{i,j,k\}$ is a some fixed permutation of $\{1,2,3\}$. If the identity (1) holds for a fixed i,j,k, then we say that a groupoid $(G,*)$ is *para – associative of type (i, j, k)* or, that it is *(i, j, k) – associative* (see [2],[3] or [12]). The condition (1) is called *the (i, j, k) – associativity*.

By *a BCI – algebra* we mean a general algebra $(G,*,0)$ of type $(2,0)$ with the following conditions:

$$((x*y)*(x*z))*(z*y)=0, \tag{2}$$

$$(x*(x*y))*y=0, \tag{3}$$

(4) $$x*x=0,$$

(5) $$x*y=y*x=0 \text{ implies } x=y,$$

(6) $$x*0=0 \text{ implies } x=0.$$

Now we prove that a para – associative BCI – algebra is a semigroup.

1) The case of the (1,2,3) – associativity is trivial.

2) The case of the (1,3,2) – associativity. In this case we have

$$(x_1*x_2)*x_3=x_1*(x_3*x_2).$$

Using this condition, we obtain

$$(x*y)*(y*x)=x*[(y*x)*y]=x*[y*(y*x)]=[x*(y*x)]*y=$$
$$=[(x*x)*y]*y=(0*y)*y=0*(y*y)=0*0=0.$$

By a similar calculation we get $(y*x)*(x*y)=0$. Hence

$$(x*y)*(y*x)=(y*x)*(x*y)=0,$$

which implies (by (5)) $x*y=y*x$. Therefore

$$(x*y)*z=x*(z*y)=x*(y*z),$$

i.e. a (1,3,2) – associative BCI – algebra is a semigroup.

3) The case of the (2,1,3) – associativity. In this case we have

$$(x_1*x_2)*x_3=x_2*(x_1*x_3),$$

which gives

$$(x*y)*(y*x)=y*[x*(y*x)]=y*[(y*x)*x]=[(y*x)*y]*x=[x*(y*y)]*x=$$
$$=(y*y)*(x*x)=0*0=0.$$

In the same manner, we prove $(y*x)*(x*y)=0$. Now from (5) immediately follows $x*y=y*x$. Hence

$$(x*y)*z=(y*x)*z=x*(y*z),$$

which proves that $(G,*,0)$ is a commutative semigroup.

4) The case of the (2,3,1) – associativity. Now we have

$$(x_1*x_2)*x_3=x_2*(x_3*x_1),$$

which together with (3) implies

$$(x*y)*(y*x)=(x*(x*y))*y=0.$$

Since x,y are arbitrary, then also $(y*x)*(x*y)=0$. Thus $x*y=y*x$ by (5). Therefore $$(x*y)*z=y*(z*x)=(z*x)*y=x*(y*z),$$

which completes the proof of this part.

5) The case of the (3,1,2) – associativity. Now a BCI – algebra $(G,*,0)$

satisfies the condition

$$(x_1*x_2)*x_3=x_3*(x_1*x_2)$$

Using (3) and para-associativity we get

$$0=[x*(x*y)]*y=y*[x*(x*y)],$$

which implies (by (5)) that $x*(x*y)=y$. Hence, putting $y=x$, we have $x=x*(x*x)=(x*x)*x$, i.e. $x=x*0=0*x$. This means that 0 is a neutral element of this BCI-algebra. Moreover this BCI-algebra is commutative. Indeed

$$x_1*x_3=(x_1*0)*x_3=x_3*(x_1*0)=x_3*x_1$$

for all $x_1, x_3 \in G$.

As it is well-known (see [6] Theorem 2) every BCI-algebra satisfies the condition

$$(((x*y)*z)*(u*z))*((x*u)*y)=0.$$

If 0 is a neutral element, then putting $z=0$ in this condition, we obtain

$$((x*y)*u)*((x*u)*y)=0,$$

which (by commutativity and (5)) implies $(x*y)*u=(x*u)*y$.
Hence

$$(x*y)*z=(x*z)*y=(z*y)*x=$$
$$x*(z*y)=x*(y*z).$$

Therefore $(G,*,0)$ is a commutative semigroup.

6) The case of the (3,2,1)-associativity. In this case the condition (1) has the form

$$(x_1*x_2)*x_3=x_3*(x_2*x_1).$$

Putting $x_1=x$, $x_2=y$ and $x_3=y*x$, we obtain

$$(x*y)*(y*x)=(y*x)*(y*x)=0$$

by (4). Putting $x_1=y$, $x_2=x$ and $x_3=x*y$, we have

$$(y*x)*(x*y)=(x*y)*(x*y)=0.$$

Hence (by (5)) $x*y=y*x$ for all $x,y \in G$.

Now from (3) follows that

$$0=(x*(x*x))*x=(x*0)*x=x*(x*0)=x*(0*x).$$

Because $(G,*,0)$ is commutative, then

$$(0*x)*x=x*(0*x)=0,$$

which proves (by (5)) that 0 is a neutral element.

As in the previous case we can prove that (x∗y)∗u=(x∗u)∗y and that the (3,2,,1)-associative BCI-algebra is asociative.

From the above results follows that every para-associative BCI-algebra is associative. But every associative BCI-algebra is a Boolen group [5].

Conversely, every group of the exponent 2 is commutative. Since Iséki proved in [8] that this group is an associative BCI-algebra, then it is also para-associative.

Therefore we obtain our main result.

Theorem. *A BCI-algebra (G, ∗, 0) is para-associative if and only if it is a group in which* $x^2=0$ *for every* $x\in G$, *where 0 is a neutral element of this group.*

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STRESZCZENIE

W tej pracy zajmujemy się para-łącznymi BCI-algebrami, tzn. BCI-algebrami (G,∗,0) z dodatkowym warunkiem $(x_1*x_2)*x_3=x_i*(x_j*x_k)$, gdzie {i,j,k} jest pewną ustaloną permutacją zbioru {1,2,3}. Dowodzimy iż BCI-algebra jest para-łączna wtedy i tylko wtedy gdy jest grupą Boole'a.