Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka VIII, Częstochowa 2000 - 2001

Analysis of Multiserver Queueing System with Restricted Summarized Volume

Oleg Tikhonenko and Konstantin Klimovich

A multiserver queueing system without waiting places is considered. The arrival process is Poisson, the joint distribution function of a demand length and its service time is arbitrary and the summarized demands volume is restricted by constant value. The steady-state distribution of the number of demands presenting in the system and probability of demand loss are obtained.

1. Introduction

We consider a multiserver queueing system with nonhomogeneous demands in which each demand may be characterized by some random length. Generally the service time of the demand depends on its length. However, the length and service time of different demands are assumed to be independent. Each demand occupies some space requirement (which is equal to its length) in a buffer during its waiting and service time and leaves the buffer immediately at the moment of finishing service. Assume, that the buffer has finite capacity, i.e. the sum of demands lengths (or summarized volume) in the buffer is restricted. The demand will be lost if there are no free space in the buffer at the moment of its arriving. It's important to obtain the steady-state distribution of the number of demands presenting in the system and loss probability.

We present the analysis of multiserver queueing system with restricted by constant value summarized volume in which the demand length and service time are dependent. (We use the methods of nonhomogeneous demands queueing theory [1].)

Consider a multiserver queueing system in which demands entrance flow forms a Poisson process with parameter a. Denote as ξ and ζ the service time and the length of the demand respectively. Let $F(x,t) = \mathbf{P}\{\zeta < x, \xi < t\}$ be the joint distribution function of ζ and ξ random variables, $L(x) = F(x,\infty)$ and $B(t) = F(\infty,t)$ are the distribution functions of

ζ and ξ random variables respectively.

Denote as $\sigma(t)$ the full sum of lengths of the demands presenting in the system at time moment t. This value is named by summarized volume [1]. Let $\eta(t)$ be the number of demands in the system at time moment t. Assume that summarized volume is restricted by some constant value $V > 0$. If $\eta(\tau - 0) = n$ or (in case of $\eta(\tau - 0) < n$) demand length is equal to x and $\sigma(\tau - 0) + x > V$, the demand arriving at time moment t will be lost at this moment and does not influence to further system's behaviour. Otherwise the demand remains in the system and $\eta(\tau) =$ $= \eta(\tau - 0) + 1, \sigma(\tau) = \sigma(\tau - 0) + x$.

This system we shall identify as $M/G/n/0(V)$.

The stability condition for this queue is $\rho = a\beta_1 < \infty$, where β_1 is the first moment of service time.

Then, $\eta(t) \Rightarrow \eta$ and $\sigma(t) \Rightarrow \sigma$ when $t \to \infty$ in the sense of a weak convergence, where η and σ is the stationary number of customers and summarized volume respectively.

We shall solve the problem of finding the stationary distribution of number of customers in the system and loss probability in general case.

2. The stationary demands number distribution and loss probability

The queueing system $M/G/n/0(V)$ is described by stochastic process

$$\left(\eta(t), \sigma(t), \xi_1^*(t), \ldots, \xi_{\eta(t)}^*(t)\right), \tag{1}$$

where $\xi_i^*(t)$ is residual service time of i-th demand in the system at time moment $t, i = \overline{1, \eta(t)}$. The numbering of the demands in the system, is considered random, i.e., the demands are numbered by one of $\eta(t)!$ equiprobable ways.

Assume, that the components $\xi_i^*(t)$ are absent if $\eta(t) = 0, \sigma(t) = 0$.

Process (1) (which, generally speaking, is not Markovian process) is characterized by functions having following probabilistic sense.

$$G_k(t, x, y_1, \ldots, y_k)dxdy_1 \ldots dy_k =$$
$$= \mathbf{P}\{\eta(t) = k, \sigma(t) \in [x; x + dx), \xi_i^*(t) \in [y_i; y_i + dy_i),$$
$$i = \overline{1, k}\}, k = \overline{1, n}; \tag{2}$$

$$P_k(t, y_1, \ldots, y_k) = \int_0^V G_k(t, x, y_1, \ldots, y_k)dx; \tag{3}$$

$$P_0(t) = \mathbf{P}\{\eta(t) = 0\}; \quad (4)$$

$$P_k(t) = \mathbf{P}\{\eta(t) = k\} = \int_0^\infty \ldots \int_0^\infty P_k(t, y_1, \ldots, y_k)dy_1 \ldots dy_k, \ k = \overline{1, n}. \quad (5)$$

If $\rho = a\beta_1 < \infty$, the next limits exist.

$$\lim_{t\to\infty} G_k(t, x, y_1, \ldots, y_k) = g_k(x, y_1, \ldots, y_k); \quad (6)$$

$$\lim_{t\to\infty} P_k(t, y_1, \ldots, y_k) = p_k(y_1, \ldots, y_k) = \int_0^V g_k(x, y_1, \ldots, y_k)dx; \quad (7)$$

$$\lim_{t\to\infty} P_k(t) = p_k = \int_0^\infty \ldots \int_0^\infty g_k(x, y_1, \ldots, y_k)dy_1 \ldots dy_k, \ k = \overline{1, n}; \quad (8)$$

$$\lim_{t\to\infty} P_0(t) = p_0.$$

Let $E_y(x)$ be a conditional distribution function of maintained demand length if residual time of its service is equal to y. As it shown in [1],

$$dE_y(x) = [1 - B(y)]^{-1} \int_{u=y}^\infty dF(x, u). \quad (9)$$

Denote as $H_y(x) = [1 - B(y)]E_y(x)$, so $dH_y(x) = \int\limits_{u=y}^{\infty} dF(x, u)$.

Theorem 1. *Functions $g_k(x, y_1, \ldots, y_k)$ for $M/G/n/0(V)$ system, describing steady conditions of the system, are presented as*

$$g_k(x, y_1, \ldots, y_k)dx = C\frac{a^k}{k!}d\left[H^*_{y_1} \ldots {}^* H_{y_k}(V)\right], \quad (10)$$

where $H_y(x) = \int\limits_{v=0}^{x}\int\limits_{u=y}^{\infty} dF(v, u)$, *$C$ is some constant value and* $H^*_{y_1} \ldots {}^* H_{y_k}(x)$ *is a Stieltjes convolution of* $H_{y_1}(x), \ldots, H_{y_k}(x)$ *functions,* $k = \overline{1, n}$.

Proof. For a simplicity we assume, that the density $f(x, y)$ of (ζ, ξ) random vector exists, though it is possible to prove the theorem without this assumption.

It is easy to show, that the gated functions (6) - (8) satisfy to the next stationary equations.

$$-\frac{\partial p_1(y)}{\partial y} =$$

$$= -a\int_0^V g_1(x,y)L(V-x)dx - ap_0\int_0^V f(u,y)du + p_2(0,y) + p_2(y,0); \quad (11)$$

$$-\sum_{i=1}^{k}\frac{\partial p_k(y_1,\dots,y_k)}{\partial y_i} = -a\int_0^V g_k(x,y_1,\dots,y_k)L(V-x)dx+$$

$$+\frac{a}{k}\sum_{i=1}^{k}\int_{x=0}^{V} g_{k-1}(x,y_1,\dots,y_{i-1},y_{i+1},\dots,y_k)\int_{u=0}^{V-x} f(u,y_i)dudx+$$

$$+\sum_{i=1}^{k+1} p_{k+1}(y_1,\dots,y_{i-1},0,y_i,\dots,y_k),\ k=\overline{2,n-1}; \quad (12)$$

$$\sum_{i=1}^{n}\frac{\partial p_n(y_1,\dots,y_n)}{\partial y_i} =$$

$$= \frac{a}{\eta}\sum_{i=1}^{n}\int_{x=0}^{V} g_{k-1}(x,y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)\int_{u=0}^{V-x} f(u,y_i)dudx; \quad (13)$$

$$p_1(0) = ap_0L(V); \quad (14)$$

$$p_k(y_1,\dots,y_{i-1},0,y_{i+1},\dots,y_k) =$$

$$= a\int_0^V g_{k-1}(x,y_1,\dots,y_{i-1},y_{i+1},\dots,y_k)L(V-x)dx,$$

$$i=\overline{1,k};\ k=\overline{2,n}. \quad (15)$$

Taking into account a symmetry of $p_k(y_1,\dots,y_k)$ functions on variables $y_1,\dots,y_k$, it is easy to show by a direct substitution, that the functions, defined by (10) relations, satisfy to equations (11) - (15).

The theorem is proved.

Corollary 1. *For $M/G/n/0(V)$ queueing system the functions $p_k(y_1,\dots,y_k)$ for $k=\overline{1,n}$ are determined by relations*

$$p_k(y_1,\dots,y_k) = \frac{Ca^k}{k!}H^*_{y_1}\dots{}^*H_{y_k}(V). \quad (16)$$

Corollary 2. *The stationary probabilities of presence k customers in a system $M/G/n/0(V)$ are determined by relation*

$$p_k = p_0\frac{a^k}{k!}R_k(V),\ k=\overline{1,n}, \quad (17)$$

where $R_k(x) = R^k_*(x), R(x) = \int\limits_{u=0}^{x}\int\limits_{y=0}^{\infty} y dF(u,y)$,

$$p_0 = \left[\sum_{k=0}^{n} \frac{a^k}{k!} R_k(V)\right]^{-1}. \tag{18}$$

The relation (17) is proved by a direct evaluation of the applicable integral in (8). The relation (18) follows from (14) and normalization conditions $\sum\limits_{k=0}^{m} p_k = 1$.

Denote as p_L the loss probability of the demand in queueing system under consideration. Apparently, the probability p_L can be determined from the next equilibrium equation

$$a(1-p_L) = \sum_{k=1}^{n} \int_0^{\infty} \ldots \int_0^{\infty} \sum_{i=0}^{k} \mu(y_i) p_k(y_1, \ldots, y_k) dy_1 \ldots dy_k,$$

where $\mu(y) = b(y)(1-B(y))^{-1}, b(y)$ is a density function of a service time. At the end, as it follows from (16), we have

$$p_L = 1 - p_0 \sum_{k=0}^{n-1} \frac{a^k}{k!} X^* R_k(V), \tag{19}$$

where $X(x) = \int\limits_0^{\infty} \frac{H_y(x)}{1-B(y)} dB(y) = \int\limits_0^{\infty} E_y(x) dB(y)$.

For $M/G/\infty(V)$ queueing system relations (17)-(19) may be represented as

$$p_k = p_0 \frac{a^k}{k!} R_k(V),\ k = 1, 2, \ldots; p_0 = \left[\sum_{k=0}^{\infty} \frac{a^k}{k!} R_k(V)\right]^{-1};$$

$$p_L = 1 - p_0 \sum_{k=0}^{\infty} \frac{a^k}{k!} X^* R_k(V).$$

Some other examples of steady-state queueing systems are discussed in [2], [3].

Conclusions

The analysis of multiserver queueing system with restricted summarized volume is presented. The stationary distribution of number of customers in the system and loss probability are obtained.

To determine the amount of memory space of computer systems with a given probability of losing messages, it will be very useful to have estimations of stationary distribution of number of customers in the system and loss probability of various queueing systems with finite memory and service time depending on the length.

References

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Oleg Tikhonenko
Pedagogical University of Częstochowa
Institute of Mathematics
Al. Armii Krajowej 13/15,
42-200 Częstochowa
E-mail: tikhon@wsp.czest.pl
and
Konstantin Klimovich
Department of Intellectual Systems
Faculty of Radiophysics and Electronics
Byelorussian State University
F. Scorina ave., 4, 220050, Minsk, Belarus
E-mail: klimovich_k@hotmail.com