Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka VIII, Częstochowa 2000 - 2001

A Note on Some Class of Locally Boolean Algebras

Andrzej Zbrzezny

In [5] J. Płonka introduced the notion of a *locally Boolean algebra* as an algebra $\mathbf{A} = \langle A, \vee, \wedge, ' \rangle$ of type $\langle 2,2,1 \rangle$ where the reduct $\langle A, \vee, \wedge \rangle$ is a distributive lattice and there exists a congruence R *of* $\mathbf{A}$ such that any congruence class $[a]_R, a \in A$ is a Boolean algebra with respect to the operations $\vee, \wedge$ and $'$ restricted to $[a]_R$. It was proved in [5] that the class of all locally Boolean algebra forms a variety. All subdirectly irreducible locally Boolean algebras were described in [6].

In this paper we consider some particular class of locally Boolean algebras defined as follows. Let U be a fixed set. An algebra $\mathbf{A} = \langle A, \vee, \wedge, ' \rangle$ of type $\langle 2,2,1 \rangle$ is said to be a *conditional set algebra over* U (*cs-algebra* for short) if and only if $A \subseteq \{\langle X, Y \rangle : Y \subseteq X \subseteq U\}$, and the operations $\vee, \wedge$ and $'$ are defined in the following way:

$$\langle X_1, Y_1 \rangle \vee \langle X_2, Y_2 \rangle = \langle X_1 \cup X_2, Y_1 \cup Y_2 \rangle$$

$$\langle X_1, Y_1 \rangle \wedge \langle X_2, Y_2 \rangle = \langle X_1 \cap X_2, Y_1 \cap Y_2 \rangle$$

$$\langle X_1, Y_1 \rangle' = \langle X_1, X_1 \backslash Y_1 \rangle,$$

for every $\langle X_1, Y_1 \rangle, \langle X_2, Y_2 \rangle \in A$.

In the case when $A = \{X, Y\rangle : Y \subseteq X \subseteq U\}$, the algebra $\mathbf{A}$ is said to be *a full cs-algebra over* U and is denoted by $fcs(U)$. Full cs–algebras were introduced (under the different name) by K.Hałkowska.

It is easy to observe [cf.3] that the class of all isomorphic images of cs–algebras does not form a variety. In the present paper we prove that the considered class forms a quasivariety.

Let us denote the clas of all cs–algebras by CS and the only (up to isomorphism) three element cs-algebra by $\mathbf{C}$. The elements of the universe of algebra $\mathbf{C}$ will be denoted by $\mathbf{-1}, \mathbf{0}$ and $\mathbf{1}$, where $\mathbf{-1}$ stands for $\langle \emptyset, \emptyset \rangle$, $\mathbf{0}$ stands for $\langle U, \emptyset \rangle$ and 1 stands for $\langle U, U \rangle$.

The operations $\vee, \wedge$ and$'$ in algebra $\mathbf{C}$ are given by the following tables:

∨	−1	0	1
−1	−1	0	1
0	0	0	1
1	1	1	1

∧	−1	0	1
−1	−1	−1	−1
0	−1	0	0
1	−1	0	1

	′
−1	−1
0	1
1	0

LEMMA. $ISP(\{\mathbf{C}\}) = I(CS)$.

Proof.

($\subseteq$). In order to show that $ISP(\{\mathbf{C}\}) \subseteq I(CS)$ we shall prove that $SP(\{\mathbf{C}\}) \subseteq I(CS)$. Let us assume that algebra $\mathbf{A} \in SP(\{\mathbf{C}\})$. It means that $\mathbf{A}$ is a subalgebra of the direct product $\prod_{t\in T} \mathbf{B}_t$, where for all $t \in T$, $\mathbf{B}_t = \mathbf{C}$. Let us define the mapping $\varphi \longrightarrow fcs(U)$ it the following way:

$$\varphi(\langle a_t\rangle_{t\in T}) = \langle \{t \in T : a_t \in \{0,1\}\}, \{t \in T : a_t = 1\}\rangle$$

It is routine to prove that φ is one–to–one mapping from $\mathbf{A}$ into $fcs(T)$.

In order to prove that φ is a homomorphism we have to show that the following three conditions hold:

(1) $\varphi(\langle a_t\rangle_{t\in T} \vee \langle b_t\rangle_{t\in T}) = \varphi(\langle a_t\rangle_{t\in T}) \vee \varphi(\langle b_t\rangle_{t\in T})$

(2) $\varphi(\langle a_t\rangle_{t\in T} \wedge \langle b_1\rangle_{t\in T}) = \varphi(\langle a_t\rangle_{t\in T}) \wedge \varphi(\langle b_t\rangle_{t\in T})$

(3) $\varphi((\langle a_t\rangle_{t\in T})') = (\varphi(\langle a_t\rangle_{t\in T}))'$

In order to prove the condition (1) note that

$$\varphi(\langle a_t\rangle_{t\in T} \vee \langle b_t\rangle_{t\in T}) =$$
$$= \varphi(\langle a_t \vee b_t\rangle_{t\in T}) = \langle \{t \in T : a_t \vee b_t \in \{0,1\}\}, \{t \in T : a_t \vee b_t = 1\}\rangle$$

and

$$\varphi(\langle a_t\rangle_{t\in T}) \vee \varphi(\langle b_t\rangle_{t\in T}) =$$
$$\langle \{t \in T : a_t \in \{0,1\}\} \cup \{t \in T : b_t \in \{0,1\}\},$$
$$\{t \in T : a_t = 1\} \cup \{t \in T : b_t = 1\}\rangle$$

Now we see that the condition (1) is equivalent to the following two identities:

(i) $\{t \in T : a_t \vee b_t \in \{0,1\}\} =$
$= \{t \in T : a_t \in \{0,1\}\} \cup \{t \in T : b_t \in \{0,1\}\}$

and

(ii) $\{t \in T : a_t \vee b_t = 1\} = \{t \in T : a_t = 1\} \cup \{t \in T : b_t = 1\}$

Using the table for the operation $\vee$ it is easy to check that these identities hold.

In order to prove the condition (2) note that

$$\varphi(\langle a_t\rangle_{t\in T} \wedge \langle b_t\rangle_{t\in T}) = \varphi(\langle a_t \wedge b_t\rangle_{t\in T}) =$$
$$= \langle\{t\in T : a_t \wedge b_t \in \{\mathbf{0},\mathbf{1}\}\}, \{t\in T : a_t \wedge b_t = \mathbf{1}\}\rangle$$

and

$$\varphi(\langle a_t\rangle_{t\in T}) \wedge \varphi(\langle b_t\rangle_{t\in T}) =$$
$$= \langle\{t\in T : a_t \in \{\mathbf{0},\mathbf{1}\}\} \cap \{t\in T : b_t \in \{\mathbf{0},\mathbf{1}\}\},$$
$$\{t\in T : a_t = \mathbf{1}\} \cap \{t\in T : b_t = \mathbf{1}\}\rangle$$

Now we see that the condition (2) is equivalent to the following two identities:

(iii) $\{t\in T : a_t \wedge b_t \in \{\mathbf{0},\mathbf{1}\}\} =$
$= \{t\in T : a_t \in \{\mathbf{0},\mathbf{1}\}\} \cap \{t\in T : b_t \in \{\mathbf{0},\mathbf{1}\}\}$

and

(iv) $\{t\in T : a_t \wedge b_t = \mathbf{1}\} = \{t\in T : a_t = \mathbf{1}\} \cap \{t\in T : B_t = \mathbf{1}\}$

Using the table for the operation $\wedge$ it is easy to check that these identities hold.

In order to prove the condition (3) note that

$$\varphi((\langle a_t\rangle_{t\in T})') = \varphi(\langle a_t'\rangle_{t\in T}) =$$
$$= \langle\{t\in T : a_t' \in \{\mathbf{0},\mathbf{1}\}\}, \{t\in T : a_t' = \mathbf{1}\}\rangle =$$

and

$$(\varphi(\langle a_t\rangle_{t\in T}))' = \langle\{t\in T : a_t \in \{\mathbf{0},\mathbf{1}\}\}, \{t\in T : a_t = \mathbf{1}\}\rangle' =$$
$$= \langle\{t\in T : a_t \in \{\mathbf{0},\mathbf{1}\}\}, \{t\in T : a_t \in \{\mathbf{0},\mathbf{1}\}\}\setminus\{t\in T : a_t = \mathbf{1}\}\rangle =$$
$$\langle\{t\in T : a_t \in \{\mathbf{0},\mathbf{1}\}\}, \{t\in T : a_t = \mathbf{0}\}\rangle$$

Now we see that the condition (3) is equivalent to the following two identities:

(v) $\{t\in T : a_t' \in \{\mathbf{0},\mathbf{1}\} = \{t\in T : a_t \in \{\mathbf{0},\mathbf{1}\}\}$

and

(vi) $\{t\in T : a_t' = \mathbf{1}\} = \{t\in T : a_t = \mathbf{0}\}$

Using the table for the operation $'$ it is easy to check that these identities hold.

We have proved that φ is one–to–one homomorphisn from $\mathbf{A}$ into $fcs(T)$ and therefore the algebra $\mathbf{A}$ is isomorphic with some subalgebra of the algebra $fcs(T)$. It means that $\mathbf{A} \in I(CS)$. so we have $SP(\{\mathbf{C}\}) \subseteq I(CS)$, and eventually $ISP(\{\mathbf{C}\}) \subseteq I(I(CS)) \subseteq I(CS)$.

($\supseteq$). In order to show that $I(CS) \subseteq ISP(\{\mathbf{C}\})$ we shall prove that $CS \subseteq ISP(\{\mathbf{C}\})$. Let us assume that $\mathbf{A} \in CS$. It means that $\mathbf{A}$ is a cs–algebra over some set U. Let us consider the U-indexed direct product $\prod_{t\in U} \mathbf{B}_t$ of the copies of $\mathbf{C}$, i.e. for all $t \in U, \mathbf{B}_t = \mathbf{C}$. Now we define the mapping $\varphi : \mathbf{A} \longrightarrow \prod_{t\in U} \mathbf{B}_t$ in the following way:

$$\psi(\langle X, Y\rangle) = \langle a_t\rangle_{t\in U}, \text{where } \forall_{t\in U} a_t = \begin{cases} \mathbf{1}, \text{ if } t \in X & \text{and} \quad t \in Y \\ \mathbf{0}, t \in X & \text{and} \quad t \notin Y \\ -\mathbf{1}, \text{ otherwise} & \end{cases}$$

It is routine to prove that ψ is one-to-one mapping from $\mathbf{A}$ into $\prod_{t\in T} \mathbf{B}_t$.

In order to prove that ψ is a homomorphism we have to show that the following three conditions hold:

(4) $\psi(\langle A, B\rangle \vee \langle C, D\rangle) = \psi(\langle A, B\rangle) \vee \psi(\langle C, D\rangle)$

(5) $\psi(\langle A, B\rangle \wedge \langle C, D\rangle) = \psi(\langle A, B\rangle) \wedge \psi(\langle C, D\rangle)$

(6) $\psi(\langle A, B\rangle') = (\psi(\langle A, B\rangle))'$

In order to prove the condition (4) one have to prove that

$$(\forall t \in T)\psi(\langle A, B\rangle \vee \langle C, D\rangle)(t) = \psi(\langle A, B\rangle)(t) \vee \psi(\langle C, D\rangle)(t)$$

which is equivalent to the following condition:

$$(\forall t \in T)(\forall w \in \{\mathbf{1}, \mathbf{0}, -\mathbf{1}\})[\psi(\langle A, B\rangle \vee \langle C, D\rangle)(t) = w \Leftrightarrow$$

$$\Leftrightarrow \psi(\langle A, B\rangle)(t) \vee \psi(\langle C, D\rangle)(t) = w]$$

Eventually, in order to prove the last condition it is sufficient to show that the following three conditions hold for every $t \in T$:

(a) $t \in A \cup C$ and $T \in B \cup D \Leftrightarrow (t \in A$ and $t \in B)$ or $(t \in A$ and $t \in B)$

(b) $t \in A \cup C$ and $T \notin B \cup D \Leftrightarrow [(t \in A$ and $t \notin B)$

and $(t \notin C$ or $t \notin D)]$ or$[(t \notin A$ or $t \notin B)$ and $(t \in C$ and $t \notin D)]$

(c) $t \notin A \cup C \Leftrightarrow (t \notin A$ and $t \notin C)$

The condition (c) is obvious. The conditions (a) and (b) are true as $B \subseteq A$ and $C \subseteq D$.

In order to prove the condition (5) one have to prove that

$$(\forall t \in T)\psi(\langle A, B\rangle \wedge \langle C, D\rangle)(t) = \psi(\langle A, B\rangle)(t) \wedge \psi(\langle C, D\rangle)(t)$$

which is equivalent to the following condition:

$$(\forall t \in T)(\forall w \in \{\mathbf{1}, \mathbf{0}, -\mathbf{1}\})[\psi(\langle A, B\rangle \wedge \langle C, D\rangle)(t) = w \Leftrightarrow$$

$$\Leftrightarrow \psi(\langle A, B\rangle)(t) \wedge \psi(\langle C, D\rangle)(t) = w]$$

Eventually, in order to prove the last condition it is sufficient to show that the following three conditions hold for every $t \in T$:

(d) $t \in A \cap C$ and $t \in B \cap D \Leftrightarrow$ ($t \in A$ and $t \in B$) and ($t \in C$ and $t \in D$)

(e) $t \in A \cap C$ and $t \notin B \cap D \Leftrightarrow$ [$t \in A$ and $t \notin B$ and $t \in C$] or [$t \in C$ and $t \notin D$ and $t \in A$]

(f) $t \notin A \cap C \Leftrightarrow$ ($t \notin A$ or $t \notin C$)

Clearly, these three conditions are obvious.

In order to prover the condition (6) one have to prove that

$$(\forall t \in T)\psi(\langle A, B\rangle')(t) = (\psi(\langle A, B\rangle)(t))'$$

which is equivalent to the following condition:

$$(\forall t \in T)(\forall w \in \{\mathbf{1}, \mathbf{0}, -\mathbf{1}\})\psi(\langle A, B\rangle')(t) = w \Leftrightarrow (\psi(\langle A, B\rangle)(t))' = w$$

Eventually, in order to prove the last condition it is sufficient to show that the following three conditions hold for every $t \in T$:

(g) $t \in A$ and $t \in A \backslash B \Leftrightarrow$ ($t \in A$ and $t \notin B$)

(h) $t \in A$ and $t \notin A \backslash B \Leftrightarrow$ ($t \in A$ and $t \notin B$)

(i) $t \notin A \Leftrightarrow t \notin A$

Clearly, these three conditions are obvious.

We have proved that ψ is one-to-one homomorphism from $\mathbf{A}$ into $\prod_{t \in T} \mathbf{B}_t$ and therefore algebra $\mathbf{A}$ is isomorphic with some subalgebra of the algebra $\prod_{t \in T} \mathbf{B}_t$. It means that $\mathbf{A} \in ISP(\{\mathbf{C}\})$. So we have $CS \subseteq ISP(\{\mathbf{C}\})$, and therefore $I(CS) \subseteq I(ISP(\{\mathbf{C}\})) \subseteq ISP(\{\mathbf{C}\})$. $\square$

Now let us recall the well known fact from universal algebra that for any finite algebra $\mathbf{A}, ISP(\{\mathbf{A}\})$ is a quasivariety. As $\mathbf{C}$ is a finite algebra, we conclude that $ISP(\{\mathbf{C}\})$ is a quasivariety and we can state the main result of our paper.

THEOREM. *The class of all isomorphic images of cs-algebras forms a quasievariety.*

In some forthcoming paper we are going to give an axiomatization of the considered quasivariety.

References

[1] S. Burris, H.P. Sankappanavar, *A course in universal algebra,* Springer Verlag, 1981

[2] K. Hałkowska, *On some generalization of Boolean algebras,* **Demonstratio Mathematica**, No 10 (3-4) (1977), 731-749

[3] K. Hałkowska, *A note on matrices for systems of nonsense-logics,* **Studia Logica**, No 48 (1989), pp. 461-464

[4] K. Hałkowska, A. Zając *On some three-valued propositional logic* (in Polish), **Acta Universitatis Wratislaviensis**, No 1017, Logika 13 (1988), pp. 41-49

[5] J. Płonka, *On bounding congruences in some algebras having the lattice structure,* **Universal Algebra and Applications, Banach Center Publications**, Volume 9, PWN, Warszawa, (1982), 203-207

[6] T. Wesołowski, *Subdirectly irreducible locally Boolean Algebras,* **Studia Scientiarum Mathematicarum Hungarica**, No 23 (1988), 257-263

[7] A. Zbrzezny, *The sequent axiomatization of some three-valued propositional logic,* **Procedings of the V Universal algebra symposium, Universal and Applied Algebra, Turawa 1988**, World Scientific, Singapore, (1989), 378-388

Institute of Mathematics and Computer Science
Pedagogical University of Częstochowa
Al. Armii Krajowej 13/15
42-200 Częstochowa