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Remarks on Geometrical Sequences of Higher Degrees

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At secondary school among geometrical and arithmetical sequences one can consider geometrical and arithmetical sequences of higher degree. Informations about arithmetical sequences of higher degree and their applications one can find in graduate textbook [4] and in the papers [1–3].

In this paper we announce some properties of geometrical sequences of higher degrees. For every sequence $\{a_n\}$ $(a_n \neq 0)$ we form the sequence $\{q_{nk}\}$ $(k \in N \setminus \{0\})$ given by the terms:

$$q_{n1} = \frac{a_{n+1}}{a_n}, \quad q_{n2} = \frac{q_{n+1,1}}{q_{n1}}, \quad \ldots, \quad q_{nk} = \frac{q_{n+1,k-1}}{q_{n,k-1}}, \quad (n \in N \setminus \{0\}).$$

Elements of each particular sequence are in the succesive columns of the following table of succesive quotients:

$$\begin{array}{c|ccccccc}
a_1 & & & & & & & \\
 & q_{11} & & & & & & \\
a_2 & & q_{12} & & & & & \\
 & q_{21} & & q_{13} & & & & \\
a_3 & & q_{22} & & q_{14} & & & \\
 & q_{31} & & q_{23} & & \ddots & & \\
a_4 & & q_{32} & \vdots & & & & \\
 & q_{41} & \vdots & & & & & \\
a_5 & \vdots & & & & & & \\
\vdots & & & & & & &
\end{array} \tag{1}$$

The sequence $\{a_n\}$ is called *geometrical sequence of the k-th degree* $(k \geq 1)$ if and only if the sequence $\{q_{nk}\}$ is constant sequence of elements different from 1. A constant sequence is called geometrical sequence of the 0-th degree. Thus an ordinary geometrical sequence with a quotient $q \neq 1$ is geometrical sequence of the first degree.

One can prove, that

Theorem 1 *If $\{a_n\}$ is a geometrical sequence of the k-th degree, then the n-th element of this sequence may be expressed by the succesive quotients as follows (see [3]):*

$$a_n = a_1^{\binom{n-1}{0}} \cdot q_{11}^{\binom{n-1}{1}} \cdot q_{12}^{\binom{n-1}{2}} \cdot \ldots \cdot q_{1k}^{\binom{n-1}{k}}, \tag{2}$$

where the combinatorial symbol $\binom{r}{s}$ has the followig meaning:

$$\binom{r}{s} = \frac{r!}{(r-s)!s!}$$

Let us notice, that in the formula (2) are only the powers in which the first elements of individual columns of the table (1) are the basics.

We can also write the formula (2) in the form:

$$a_n = a_1 \cdot \prod_{i=1}^{k} q_{1i}^{\binom{n-1}{i}} \tag{3}$$

The product of n initial elements of geometrical sequence $\{a_n\}$ of the k-th degree has the form:

$$\pi_n = a_1^n \cdot q_{11}^{\binom{n}{2}} \cdot q_{12}^{\binom{n}{3}} \cdot \ldots \cdot q_{1k}^{\binom{n}{k+1}}$$

that is

$$\pi_n = a_1^n \cdot \prod_{i=1}^{k} q_{1i}^{\binom{n}{i+1}} \tag{4}$$

The formulae (3) and (4) one can prove by induction.

Example 1 *The sequence $\{a_n\}$ defined as follows*

$$a_n = r^{c_s n^s + c_{s-1} n^{s-1} + \ldots + c_2 n^2 + c_1 n + c_0} \tag{5}$$

is geometrical sequence of the s-th degree. The sequence $\{q_{ns}\}$ for it is constant sequence given by the formula:

$$q_{ns} = c_s \cdot s! \quad (n \in N \setminus \{0\}). \tag{6}$$

Let $d^\circ(W_i)$ denotes degree of polynomial $W_i(n)$ of variable n. For polynomials $W_1(n), W_2(n), \ldots, W_l(n)$ we adopt the following symbol:

$$p = \max(d^\circ(W_1), d^\circ(W_2), \ \ldots \ , d^\circ(W_l)). \tag{7}$$

Theorem 2 *If the sequence $\{a_n\}$ has the form:*

$$a_n = r_0 \cdot r_1^{W_1(n)} \cdot r_2^{W_2(n)} \cdot \ldots \cdot r_l^{W_l(n)}, \tag{8}$$

then it is geometrical sequence of the p-th degree, where p is defined by the formula (7).

Example 2 *The number a_n of all Boolean functions of n variables is described by the formula:*

$$a_n = 2^{n^2} \tag{9}$$

Hence, the sequence $\{a_n\}$ is geometrical sequence of the second degree.

Example 3 *The number a_n of binary relations of certain types defined in the set $A = \{1, 2, \ldots, n\}$ is the n-th element of a geometrical sequence of the second degree. It holds in the case of the next relations:*

reflexive: $(\forall_{a\in A}(aRa))$, $\quad a_n = 2^{2\binom{n}{2}}$,

antireflexive: $(\forall_{a\in A}[\neg(aRa)])$, $\quad a_n = 2^{2\binom{n}{2}}$,

symmetrical: $(\forall_{a,b\in A}(aRb \Rightarrow bRa))$, $\quad a_n = 2^{\binom{n+1}{2}}$,

asymmetrical: $(\forall_{a,b\in A}[aRb \Rightarrow \neg(bRa)])$, $\quad a_n = 3^{\binom{n}{2}}$,

antisymmetrical: $(\forall_{a,b\in A}(aRb \wedge bRa \Rightarrow b = a))$, $\quad a_n = 2^n \cdot 3^{\binom{n}{2}}$.

References

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