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The Mathematics Around the Golden Ratio

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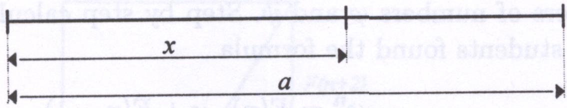
The paper gives an outline of the project of several seminars which is the subject matter of the course titled Selected Parts of Mathematics for students of Teaching Profession for Middle School Teacher (pupils from the age of 12 to 16). The students solved many different problems and they could apply basic methods of mathematical investigation - self-discovery of new knowledge, the formulation of hypotheses and their proofs - within the subject matter, which is not too abstract. Another benefit of this subject was its comprehensiveness, the subject matter involved in different mathematical disciplines.

1. According to the definition of the „golden ratio", we first derive that the ratio of length $a : x = \varphi$ is the root of the quadratic equation $x^2 - x - 1 = 0$.

$$\frac{a}{x} = \frac{x}{a-x}$$



If we designate moreover the second (negative) root of this equation by ψ, we have then

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \psi = \frac{1-\sqrt{5}}{2},$$

and e.g. the formulas are thus valid

$$\varphi^2 = \varphi + 1 \qquad \psi^2 = \psi + 1 \qquad \varphi = 1 + \tfrac{1}{\varphi} \qquad \psi = 1 + \tfrac{1}{\psi}$$
$$\varphi \cdot \psi = -1 \qquad \varphi + \psi = 1 \qquad \varphi - \psi = \sqrt{5} \quad \text{and so on.}$$

2. Another set of problems was devoted to the geometrical connections, for example to

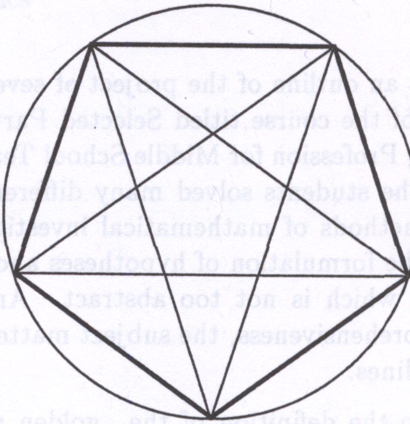
- golden ratio in the regular pentagon

- Euclidean construction of sides of the regular pentagon and decagon

$$a_5 = r \cdot \sqrt{\frac{5}{\varphi}} \qquad a_{10} = \frac{r}{\varphi}$$

- expression of the values of trigonometric functions of some angles, e.g.

$$\sin\frac{\pi}{10} = \frac{1}{2}\cdot\frac{1}{\varphi} \qquad \cos\frac{\pi}{5} = \frac{1}{2}\cdot\varphi$$



3. The intermediate phase to other problems was the investigation of the powers of numbers φ and ψ. Step by step calculation of $\varphi^2, \varphi^3, \varphi^4 \dots$ etc. the students found the formula

$$\varphi^n = F(n)\cdot\varphi + F(n-1),$$

where $F(n)$ is the n-th member of the Fibonacci sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\dots$$

and they proved this formula by induction.
In a similar way they worked for example with formulas:

$$\psi^n = -F(n)\cdot\varphi + F(n+1)$$

$$\varphi^n + \psi^n = F(n-1) + F(n+1)$$

The members of the latter sequence $\varphi^n + \psi^n$ first led the students to the concept of the general Fibonacci sequence.

4. The following large group is devoted to properties of members of the Fibonacci sequence.

The students found and proved „simple" theorems such as

$$F(n-1)\cdot F(n+1) = F^2(n) + (-1)^n$$

$$F(n-2)\cdot F(n+2) = F^2(n) - (-1)^n$$

with conclusion $F(n-2)\cdot F(n-1)\cdot F(n+1)\cdot F(n+2) = F^4(n) - 1$ or the addition theorems such as

$$\sum_{i=1}^{n} F(i) = F(n+2) - 1$$

$$\sum_{i=1}^{n} F^2(i) = F(n)\cdot F(n+1)$$

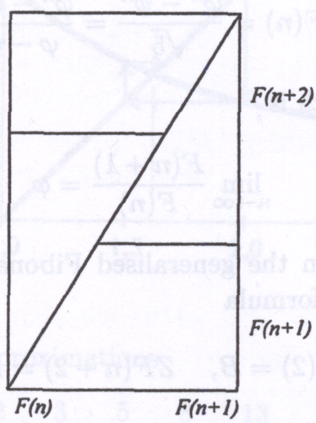
as well as theorems with more difficult to prove such as
$F^2(n) + F^2(n+1) = F(2n+1) \wedge F(n+1)\cdot(F(n) + F(n+2)) = F(2n+2)$.

The students got familiar with the well-known „paradox of missing square" which is connected with the formula

$$F(n)\cdot F(n+3) - F(n+1)\cdot F(n+2) = (-1)^{n+1}.$$

This problem led students to the question what is the limit value of the expression

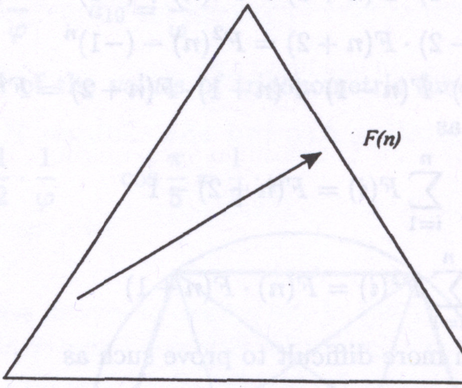
$$\frac{F(n+1)}{F(n)}.$$



At the end of this activity the students proved the formulas expressing members $F(n)$ by means of the sum of the numbers in the Pascal triangle:

$$F(2k+1) = \binom{2k}{0} + \binom{2k-1}{1} + \cdots + \binom{k}{k}$$

$$F(2k) = \binom{2k-1}{0} + \binom{2k-2}{1} + \ldots + \binom{k}{k-1}$$



5. In the following part the students worked on an explicit expression of the n-th member of the Fibonacci sequence. Assumpting that this member is of the form

$$F(n) = c_1 \cdot q_1^n + c_2 \cdot q_2^n$$

they derived the well-known formula

$$F(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}} = \frac{\varphi^n - \psi^n}{\varphi - \psi}$$

the consequence of which is

$$\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \varphi.$$

The students worked on the generalised Fibonacci sequences defined by means of the recurrent formula

$$ZF(1) = A, \quad ZF(2) = B, \quad ZF(n+2) = ZF(n+1) + ZF(n)$$

and found the explicit expression

$$ZF(n) = c_1 \cdot \varphi^n + c_2 \cdot \psi^n$$

where

$$c_1 = \frac{B - A \cdot \psi}{\varphi \cdot (\varphi - \psi)}, \quad c_2 = \frac{B - A \cdot \varphi}{\psi \cdot (\psi - \varphi)},$$

the connection with the basic Fibonacci sequence

$$ZF(n+2) = A \cdot F(n) + B \cdot F(n+1),$$

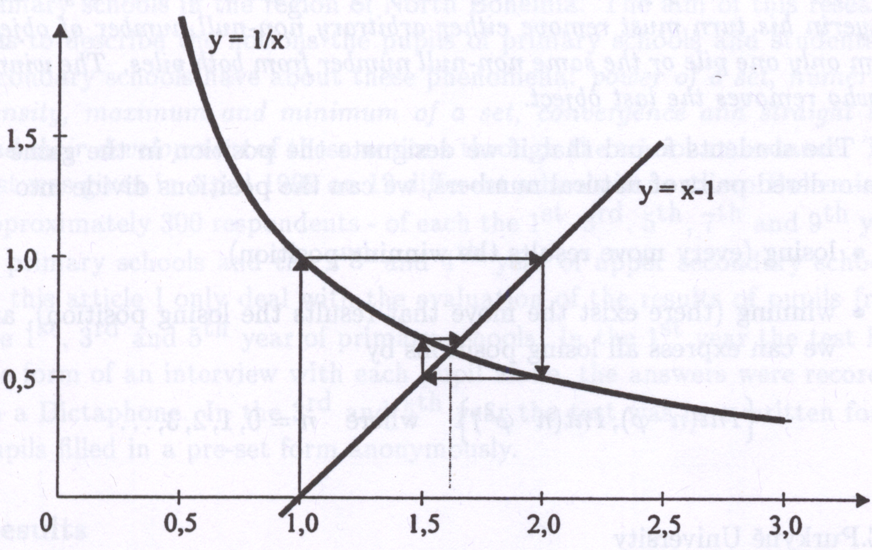
and similar limit theorem as in the first case.
The generalisation of the theorems from part 4 was also interesting.

6. Next part was devoted to the rational aproximations of the numbers φ and ψ. It started by using of the formula

$$\varphi = 1 + \frac{1}{\varphi}$$

to express the ratio φ by means of the chain fraction and the connection with the iteration process

$$x(1) = 1, \quad x(n+1) = 1 + \frac{1}{x(n)}.$$



Gradually they came to approximations

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots$$

and proved the relation $x(n) = \frac{F(n+1)}{F(n)}$.
Using the formula $x^2 = x + 1$ the students came to the iteration process

$$x(n+1) = \sqrt{1 + x(n)}$$

e.g. with initial conditions $x(1) = 1, x(1) = 2$, respectively.
The most interesting problem was the procedure based on the Newton's method with the higher velocity of the convergence. Here the students proved that if

$$x(1) = \frac{2}{1} \quad \text{and} \quad x(n+1) = \frac{x^2(n)+1}{2x(n)-1}$$

then

$$x(n) = \frac{F(2^n+1)}{F(2^n)}.$$

7. The last part was devoted to the discovery of the strategy of the special NIM-game which consists in removing objects. Its rules are as follows:

Two players take turns in removing objects from two piles. The number of objects in every pile at the beginning of the game is arbitrary. Each playerin his turn must remove either arbitrary non-null number of objects from only one pile or the same non-null number from both piles. The winner is who removes the last object.

The students found that if we designate the position in the game by non-ordered pairs of natural numbers, we can the positions divide into

- losing (every move results the winning position)
- winning (there exist the move that results the losing position), and we can express all losing positions by

$$\left\{ Int(n \cdot \varphi), Int(n \cdot \varphi^2) \right\} \quad \text{where} \quad n = 0, 1, 2, 3, \ldots$$

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