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How to Penetrate into Mathematics Actively

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At present we prefer to teach mathematics using methods which enable our students to be active. One of the methods which allow our students to be more active is investigation. In this article we wish to explain the investigative approach to teaching mathematics and to demonstrate it with the help of a non-traditional and fascinating topic - the **Fibonacci sequence.** We will also make several methodological remarks about this approach.

If we allow a certain amount of imprecision, we feel the following is a good definition of investigation in school mathematics.

A student can **investigate** if he or she is presented with a „situation" which is interesting to him/her and also appropriate to his/her ability and knowledge. The student can explore this situation and this exploration can lead to „new discoveries", which can have the form of questions and/or conjectures. (The discoveries made by the students are sometimes surprising even to the teacher). Later the student can choose some of these questions or conjectures and try to find solutions (i.e. an answer to a question or a proof or counterexample to a conjecture; if a proof of a conjecture is too difficult for students the conjecture can be only tested). Also the student can attempt to find a general method of solution - an algorithm.

When we investigate a situation and create problems or conjectures, we usually begin by accepting the „givens" in the situation, and only later challenge the „givens". This second step is a starting point investigations that modify the „givens". Very often it is only when we look at something, not as it „is", that we see its essence or significance. In both cases, there are usually many starting points for investigations.

Fibonacci sequence is the sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89, \ldots \qquad (1)$$

The numbers in this sequence are called **Fibonacci numbers.** The sequence was investigated originally by Fibonacci an Italian mathematician of the thirteenth century[1].

We use the letter F for the Fibonacci numbers in this way:

$$F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5.$$

Strategy ,,accepting the given"

1) The next terms of Fibonacci sequence

Write down next three terms of the Fibonacci sequence.
(144, 233, 377)

2) Generating the Fibonacci sequence

Try to write a formula for generation of Fibonacci sequence.

1, 1 are the first two terms, and the sum of any two consecutive terms gives the next one

$$F_n = F_{n-2} + F_{n-1}, \quad \text{where} \quad n \geq 3. \tag{2}$$

3) The Sums of the Fibonacci numbers

Investigate the sums of some of the first Fibonacci numbers, for example

$$1 + 1 + 2 + 3 = 7.$$

Solution:

Experimentation:

$F_1 + F_2 = 1 + 1 = 2$

$F_1 + F_2 + F_3 = 1 + 1 + 2 = 4$

$F_1 + F_2 + F_3 + F_4 = 1 + 1 + 2 + 3 = 7$

$F_1 + F_2 + F_3 + F_4 + F_5 = 1 + 1 + 2 + 3 + 5 = 12$

$F_1 + F_2 + F_3 + F_4 + F_5 + F_6 = 1 + 1 + 2 + 3 + 5 + 8 = 20$

Can you see any formula ? Hint: Compare the sums with the Fibonacci numbers.

Experimentation:

$F_1 + F_2 = 1 + 1 = 2 = F_4 - 1$

$F_1 + F_2 + F_3 = 1 + 1 + 2 = 4 = F_5 - 4 - 1$

[1]His full name was Leonardo of Pisa, or Leonardo Pisano in Italian since he was born in Pisa about 1175 AD. He called himself Fibonacci (short for filius Bonacci) which means son of Bonacci. He wrote 5 mathematical works, 4 books and one manuscript preserved as a letter. In one book he introduces a problem:

a pair of rabbits are put in a field and, if rabbits take a month to become mature and then produce a new pair every month after that, how many pairs will there be in twelve months time?

The answer involves the sequence (1), but it was French mathematician Edouard Lucas (1842-91) who gave the name Fibonacci numbers to this sequence.

$F_1 + F_2 + F_3 + F_4 = 1 + 1 + 2 + 3 = 7 = F_6 - 1$
$F_1 + F_2 + F_3 + F_4 + F_5 = 1 + 1 + 2 + 3 + 5 = 12 = F_7 - 1$
$F_1 + F_2 + F_3 + F_4 + F_5 + F_6 = 1 + 1 + 2 + 3 + 5 + 8 = 20 = F_8 - 1$
Also $F_1 = 1 = F_3 - 1$

Conjecture: For all natural n it is true that

$$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1 \quad (3)$$

Proof of the conjecture: We give only an outline of the proof - students should be encouraged to write a complete and grammatical argument. We rewrite formula (2) so each F_i is written as the difference of the two succeeding terms.

$$\begin{aligned} F_1 &= F_3 - F_2 \\ F_2 &= F_4 - F_3 \\ F_3 &= F_5 - F_4 \\ &\ldots\ldots\ldots \\ F_{n-1} &= F_{n+1} - F_n \\ F_n &= F_{n+2} - F_{n+1} \\ \hline F_1 + F_2 + F_3 + \ldots + F_n &= F_{n+2} - 1 \end{aligned}$$

Adding up the left-hand sides and the right-hand sides of the equations completes the proof, so we can rename our conjecture as **Theorem 1.**

In a similar way you can investigate the other kinds of the sums of Fibonacci numbers.

4) The sums of the Fibonacci numbers in the odd positions

Investigate the sums of some of the first Fibonacci numbers in the odd positions, for example

$$1 + 2 + 5 + 13 = 21$$

Conjecture (after a proof, we can label it **Theorem 2**): For all natural n it is true that

$$F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n} \quad (4)$$

5) The sums of the Fibonacci numbers in the even positions

Investigate the sums of some of the first Fibonacci numbers in the even positions, for example

$$1 + 3 + 8 = 12$$

Conjecture (after a proof, we can label it **Theorem 3**): For all natural n it is true that

$$F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1. \quad (5)$$

With the help of formulas (4) and (5) you can deduce the following theorem. Do so.

Theorem 4: For all natural numbers n it is true that

$$F_1 - F_2 + F_3 - F_4 + \ldots - F_{2n} = F_{2n} - F_{2n+1} + 1. \quad (6)$$

With the help of formula (6) we can obtain the following theorem. Do so.

Theorem 5: For all natural numbers n it is true that

$$F_1 - F_2 + F_3 - F_4 + \ldots - F_{2n} + F_{2n+1} = F_{2n} + 1.$$

6) The Fibonacci Spiral Diagram

Investigate the sums of squares of some of the first Fibonacci numbers, for example

$$1^2 + 1^2 + 2^2 + 3^2 = 15 = 3 \cdot 5$$

Experimentation:

$1^2 + 1^2 = 2 = 1 \cdot 2$
$1^2 + 1^2 + 2^2 = 6 = 2 \cdot 3$
$1^2 + 1^2 + 2^2 + 3^2 = 15 = 3 \cdot 5$
$1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 40 = 5 \cdot 8$
$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 104 = 8 \cdot 13$
Also $1^2 = 1 = 1 \cdot 1$

Conjecture (after a proof we can label it **Theorem 6**): For all natural n it is true that

$$F_1{}^2 + F_2{}^2 + F_3{}^2 + \ldots + F_n{}^2 = F_n \cdot F_{n+1} \quad (7)$$

Proof: To prove it you can use the following formula:

$$F_k \cdot F_{k+1} - F_{k-1} \cdot F_k = F_k(F_{k+1} - F_{k-1}) = F_k{}^2 \quad \text{for every natural } k > 1.$$

We can also model equality (7) as shown in figure 1. This diagram is called **the Fibonacci spiral diagram**. We can easily see, that if the sides of squares are Fibonacci numbers, then wherever we stop, we will always get a rectangle.

Note: Our conjecture can be discovered with the help of Fibonacci spiral diagram.

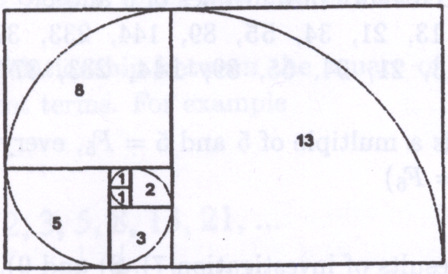

Fig.1

Remark: Investigations 3), 4), 5), 6) and also theorems 3, 4, 5 create cluster of problems which are internaly connected and with their help we can demonstrate a method of generated problems (see Kopka (1999)). Here is a brief description of this method: The student is presented with problem and assisted in solving this problem as far as necessary. In this way he is given a basis for further work. After a problem has been completely solved and clarified the teacher and the students ask further questions and generate problems which relate to the problem just solved. Thus the original problem acts to generate other problems and so we will call it a **generator problem**. The set of generated problems together with generator we can call **set (cluster) of generated problems**. As a generator problem we can take investigation 3. If a student understands investigation 3) and the proof of theorem 1, then there is a great probability, that he or she will be able (with a little help) to discover other theorems and their proofs.

Now we investigate **factors of Fibonacci numbers.**

In particular we study which terms of the sequence are divisible by terms appearing earlier.

7) There are numbers in the sequence that are divisible by two. Investigate the occurrence of even numbers in the sequence. (1, 1, **2**, 3, 5, **8**, 13, 21, **34**, 55, 89, **144**, 233, 377, **610**, 987, ...

Every third number is a multiple of 2. Notice that $2 = F_3$)

8) Investigate the occurrence of multiples of 3.

(1, 1, 2, **3**, 5, 8, 13, **21**, 34, 55, 89, **144**, 233, 377, 610, **987**, ...

Every fourth number is a multiple of 3. Notice that $3 = F_4$)

9) Investigate occurrence of multiples of 5 and 8.
(1, 1, 2, 3, **5**, 8, 13, 21, 34, **55**, 89, 144, 233, 377, **610**, 987, ...
1, 1, 2, 3, 5, **8**, 13, 21, 34, 55, 89, **144**, 233, 377, 610, 987, 1597, **2584** ...
Every fifth number is a multiple of 5 and $5 = F_5$, every sixth number is a multiple of 8 and $8 = F_6$)

10) Generalize results of investigation 7), 8) and 9).
(We can conjecture that
Conjecture: For every Fibonacci number F_k it is true that F_k divides the Fibonacci numbers in locations $2k$, $3k$, $4k$, ..., nk, ...)

Next we shall investigate relations between consecutive numbers.

11) **The difference between two consecutive terms**
Investigate the difference between consecutive terms.
(Experimentation: $F_2 - F_1 = 0$, $F_3 - F_2 = 1$, $F_4 - F_3 = 1$, $F_5 - F_4 = 2$, $F_6 - F_5 = 3$, 5, 8, 13, 21, ...
Conjecture: These differences form another Fibonacci sequence with 0 instead of 1 as the first term.)
Investigate the differences between two terms separated by one term, then by two terms, etc.

12) **The sequence of ratios of consecutive terms**
Create ratios of consecutive terms (divide each number by its successor to obtain the ratios). Does the sequence of ratios approach some specific value ?
(Experimentation: $1/1 = 1$, $1/2 = 0.5$, $2/3 = 0.666\ldots$, $3/5 = 0.600$, $5/8 = 0.625\ldots$, $8/13 = 0.615\ldots$, $13/21 = 0.619\ldots$, $21/34 = 0.618\ldots$, ...
Conjecture: The sequence of ratios of succeeding terms approaches 0.618 (to three decimal places; this irrational number is called the **Golden Ratio** or **Golden**

Number[2]**).**

13) The square of any term and the product of its adjacent terms

Investigate the relationship between the square of any term and the product of its adjacent terms. For example

1, 1, 2, 3, 5, 8, 13, 21, ...

64

(-) 65

-1

(Experimentation: $1^2 = 1 \cdot 2 - 1$, $2^2 = 1 \cdot 3 + 1$, $3^2 = 2 \cdot 5 - 1$, $5^2 = 3 \cdot 8 + 1$, $13^2 = 8 \cdot 21 + 1$, $21^2 = 13 \cdot 34 - 1$, ...)

Conjecture: The square of any term differs by one from the product of the two terms preceding and following this term Another possible formulation: For all natural $n > 1$ it is true that

$$F_n^2 = F_{n-1} \cdot F_{n+1} + (-1)^{n+1}. \tag{8}$$

Proof: We use mathematical induction

1. If $n = 2$, then we get $F_2^2 = F_1 \cdot F_3 - 1 = 1 \cdot 2 - 1$ which is true.
2. We assume that formula (8) is true for any specific n.

We must prove that the formula (8) is true also for $n + 1$.

[2] If we solve quadratic equation $x^2 - x - 1 = 0(*)$, we get two solutions Phi $= 1/2 + \sqrt{5}/2$ and -phi $= 1/2 - \sqrt{5}/2$. Number phi is our number. We can define both the golden numbers Phi and phi by equation (*). The decimal value of phi is 0.6180339887 (to 10 decimal places). The value of Phi is almost the same but begins with 1.6.. instead of 0.6.. . Euclid wrote the Elements which is a collection of 13 books on Geometry. In book 6 he shows how to find the golden section point G on the line. If you take a segment of length one and break it up into segments of lengths g and 1-g so that $g/1 = (1-g)/g$ then each of these fractions will be the golden ratio. A rectangle with such dimensions is called a golden rectangle.
The ratio of the width to length of the Parthenon in Greece approximates the golden ratio, David's ,,belly button" in the statue by Michelangelo is at a height of approximately 0.618 of his total height. Many artists find quantities in the Golden Ratio to be in an especially attractive arrangement.

$$
\begin{aligned}
F_n^2 &= F_{n-1}\cdot F_{n+1} + (-1)^{n+1} \quad \text{assumption} \\
+F_n\cdot F_{n+1} & \quad +F_n\cdot F_{n+1} \\
F_n(F_n + F_{n+1}) &= F_{n+1}(F_{n-1} + F_n) + (-1)^{n+1} \\
F_n \cdot F_{n+2} &= F_{n+1}^2 + (-1)^{n+1} \\
F_{n+1}^2 &= F_n\cdot F_{n+2} + (-1)^n \quad \text{consequent)}
\end{aligned}
$$

14) The product of two adjacent terms and product of the two terms preceding and following these terms

Investigate the relationship between the product of two adjacent terms and the product of the two terms preceding and following them. For example:

1, 1, 2, 3, 5, 8, 13, 21, ...

104

(-) 105

-1

(Exprimentation: $1\cdot 2 = 1\cdot 3 - 1$, $2\cdot 3 = 1\cdot 5 + 1$, $3\cdot 5 = 2\cdot 8 - 1$, $5\cdot 8 = 3\cdot 13 + 1$, $8\cdot 13 = 5\cdot 21 - 1, \ldots$)

Conjecture: The product of two adjacent terms differs by one from the product of the two terms preceding and following these terms.

15. The product of consecutive terms

Investigate the relationship between the products of consecutive terms. (The new sequence is $F_1\cdot F_2 = 1$, $F_2\cdot F_3 = 2$, 6, 15, 40, 104, ... The difference of two consecutive terms is always a square number).

Remark about success of investigation.
We want to remind you that not every idea is fruitful and you shouldn't feel bad if you make some „wrong" choices in the things to investigate.

Let us now move to our second strategy - the „What if not" strategy.

The strategy of not accepting the given

How would you describe the Fibonacci sequence ? Here is a list of some possible responses (see Brown (1990)):

a) The sequence starts with two given numbers.

b) The starting numbers are the same.

c) The same number is 1.

d) We do something to any two consecutive numbers to get the next number.

e) The something we do is an operation.

f) The operation is addition.

These are six important statements (attributes) about how a Fibonacci sequence is formed. We will take each statement in turn and ask the question ,,What if not ?" We must say that these statements are not independent. If we, for example, change statement b), then c) be also changed.

Statement a) The sequence starts with two given numbers. *What if not* with two given numbers ?

Possible alternatives:

a_1) Start with one given number.

a_2) Start with three given numbers.

(If, for example, we start with three given numbers, then we need perhaps another rule. In some literature, the sequence obtained by starting with three numbers and adding up triples to get the next number is called a **Tri-bonacci** sequence).

Statement b) The starting numbers are the same. *What if not* the numbers are the same ?

Possible alternative:

b_1) Consider two different starting numbers.

Statement c) *What if not* 1 ?

c_1) Consider the number 2.

c_2) Consider the number 6.

Statement d) *What if not* with two consecutive numbers ?

d_1) Consider one number.

d_2) Consider three consecutive numbers.

(The alternative d_2 is strongly connected with alternative a_2. See note after a_2).

Statement e) *What if not* an operation?

e_1) Consider relation such as $<$.

(In this case the sequence need not be unique. Rule: $T_n < T_{n-2}+T_{n-1}$)

Statement f) *What if not* addition ?

f_1) Consider subtraction.

f_2) Consider multiplication.

f_3) Consider division.

(It would be very interesting to investigate these alternatives).

We have taken six statements and have generated „What if not ?" alternatives. What can we do with this list ? We can ask questions. Let us demonstrate with the help of alternative d_2) to statement d, alternative c_1 to statement c) and alternative b_1) to statement b.

Alternative d_2 to d

If the first three numbers were 1, 1, 1 we have the following **Tribonacci sequence** (we now adding up three numbers to get the next):

$$1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193 \dots$$

Alternative c_1 to c

If the first two terms were 2, 2 we have the following sequence (now we are back to adding two numbers to get the next one):

$$2, 2, 4, 6, 10, 16, 26, 42, 68, 110, 178, \dots \qquad (9)$$

Alternative b_1 to b

If the first two terms were different, what might they be ? Let us take numbers 3 and 2 as the first two terms. We thus have the following sequence:

$$3, 2, 5, 7, 12, 19, 31, 50, 81, 131, 212, \dots \qquad (10)$$

Sequences (9) and (10) are examples of so-called pseudo-Fibonacci sequences. A sequence is called **pseudo-Fibonacci** if the first two terms are the same or different and we add two consecutive terms to get the next one.

16) **The pseudo-Fibonacci sequences**

Construct several other pseudo-Fibonacci sequences.

Note: General forms of pseudo-Fibonacci sequences

If the first two terms of pseudo-Fibonacci sequence are both a, we get

$$a, a, 2a, 3a, 5a, 8a, 13a, 21a, 34a, \dots \qquad (11)$$

If the first two terms of pseudo-Fibonacci sequence are a and b, we get

$$a, b, a+b, a+2b, 2a+3b, 3a+5b, 5a+8b, \dots \qquad (12)$$

Let us now investigate sequence (9) and sequence (10).

17) **The difference between any two consecutive terms**
Investigate the sequence of differences between consecutive terms of sequence (9). (We obtain 0, 2, 2, 4, 6, 10, 16, 26, ...
Conjecture: The difference generates a sequence similar to sequence (9), but with 0 instead of 2 as the first term).
Compare this with the result of investigation in 11). Investigate the differences between any two consecutive terms of sequence (10) and any other pseudo-Fibonacci sequence.
Try to generalize your discovery.
(**Conjecture**: The sequence of differences is always the same as the original pseudo-Fibonacci sequence except that the first term changes.
Explanation: From sequence (11), we see that the sequence of differences is

$$0,\ a,\ a,\ 2a,\ 3a,\ 5a,\ 8a,\ 13a,\ \ldots$$

(The first term is always 0).

From sequence (12), we see that the sequence of differences is

$$b-a,\ a,\ b,\ a+b,\ a+2b,\ 2a+3b,\ 3a+5b,\ \ldots$$

(The first term is always b - a)).

18) **The sequence of ratios of consecutive terms**
Does the sequence of ratios of consecutive terms of sequence (9) approach to any specific value ?
(If we take ratios of succeeding terms, we get the following: $2/2 = 1$, $2/4 = 0.5$, $4/6 = 0.666\ldots$, $6/10 = 0.6$, $10/16 = 0.625$, $16/26 = 0.615\ldots$, $26/42 = 0.619\ldots$, $42/68 = 0.617\ldots$, $0.612\ldots$, ... It is reasonable to conjecture that we are once more approaching the golden ratio.
Is it so and if so, why ?
If we start with sequence (11), we get $a/a = 1$, $a/2a = 0.5$, $2a/3a = 2/3$, ... It is the same sequence as in investigation 12)).
Investigate the sequence of ratios of consecutive terms of sequence (12) and several other pseudo-Fibonacci sequences.
Try to generalize your discovery.
(**Conjecture**: For all pseudo-Fibonacci sequences we approach the golden ratio, when we form the sequence of ratios of consecutive terms. To prove this conjecture, we can use sequences (11) and (12)).

19) **The square of any term and the product of its adjacent terms**
Investigate the relationship between the square of any term and the product of its adjacent terms in sequence (9). For example

$$2, 2, 4, 6, 10, 16, 26, 42, 68, 110, 178, \ldots$$

$$\begin{array}{r} 36 \\ (-)\ \ 40 \\ \hline -4 \end{array}$$

$6^2 = 4 \cdot 10 - 4$. It seems that number 4 might play the same role as the number 1 in the Fibonacci sequence. Try a few more cases and make a conjecture.

Investigate the relationship between the square of any term and the product of its adjacent terms in sequence (10) and in some other pseudo-Fibonacci sequences.

Try to generalize your discovery.

(Conjecture: In any pseudo-Fibonacci sequence this difference is always the same.

Result: The difference between the square of any term and the product of its adjacent terms in sequence (11) is a^2 or $-a^2$ and in sequence (12) is $b^2 - ab - a^2$ or $a^2 + ab - b^2$.)

20) The product of two consecutive terms and the product of the two terms preceding and following them

Investigate the relationship between the product of two consecutive terms and product of the two terms preceding and following these terms in sequence (9). For example:

$$2, 2, 4, 6, 10, 16, 26, 42, 68, \ldots$$

$$\begin{array}{r} 60 \\ (-)\ \ 64 \\ \hline -4 \end{array}$$

6.10 = 4.16 - 4. We have the same difference as in investigation 19). Try a few more cases.

Investigate the relationship between the product of two consecutive terms and product of the two terms preceding and following these terms in sequence (10) and in some other pseudo-Fibonacci sequences.

Try to generalize your discovery.

(Conjecture: In any pseudo-Fibonacci sequence this difference is always the same.
Result: The difference between the product of any two consecutive terms and the two terms preceding and following these terms in sequence (11) is a^2 or $-a^2$ and in sequence (12) is $b^2 - ab - a^2$ or $a^2 + ab - b^2$).

Remark about the „What if not?" strategy There are four major stages in the strategy

1. Choose a starting point.

2. List the statements.

3. Apply „What-if-not?" to each statement.

4. Ask questions arising from „What-if-not?"; that is, pose problems.

This article demonstrates the method of **guided investigation.** This investigation consists of several **mini-investigations.** Every point represents one mini-investigation. In each mini-investigation we give the students a „direction" to their investigation. This direction can be given by a particular example (see investigation 20)). The method of mini-investigation is an especially useful tool for school application. One particular advantage of mini-investigations in a school situation is that they do not take much time.

Investigation is only one of many teaching methods and we must create an educational unit by combining several different ones. This method was very interesting for the students and most of them were fully absorbed by the investigation.

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