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Calculus Infinitesimalis

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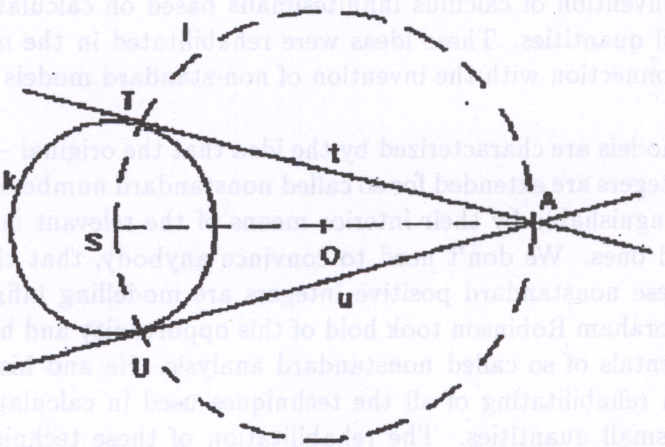
At our university, except a course of traditional mathematical analysis so called ε, δ- analysis, there has been in progress an optional course of mathematical analysis leant on main ideas of Newton and Leibnitz which lead to the invention of calculus infinitesimalis based on calculations with infinite small quantities. These ideas were rehabilitated in the mid – 20^{th} century in connection with the invention of non-standard models of the set theory.

These models are characterized by the idea that the original – standard – positive integers are extended for so called nonstandard numbers, but they are not distinguishable by their interior means of the relevant model from the standard ones. We don't need to convince anybody, that the inverse values of these nonstandard positive integers are modelling infinite small numbers. Abraham Robinson took hold of this opportunity and he built up the fundamentals of so called nonstandard analysis. He and his followers succeeded in rehabilitating of all the techniques used in calculations with the infinite small quantities. The rehabilitation of these techniques used in calculations with infinite small quantities doesn't mean rehabilitation of an original intuition needful for manipulating with these quantities. The nonstandard analysis namely didn't affect the infinite small quantities in their original form, but just modeled them owing to this, the original intuition was replaced by another intuition, else connexion, but which might be adopted just by someone who knows the nonstandard models of the set theory. But it is necessary to have a lot of knowledge in modern set theory and mathematical logic. This is the reason why it is not possible to give lectures in nonstandard analysis in common university courses.

The mentioned objection might be removed only in case if the original calculus infinitesimalis is rehabilitated in its not biased form; so not only techniques of calculations with infinity small quantities, but also the original intuition needful for manipulating with these quantities. This particular conceived rehabilitation of calculus infinitesimalis can be found in a book called ,,Calculus Infinitesimalis" written by Petr Vopěnka. One of the direction indicators of this approach is naturally nonstandard analysis and nonstandard models of the set theory at all. The second direction indicator of this approach is the alternative set theory.

In the picture there is taken down a known construction of tangents of the given circle k (with the centre S) going through the given point A lying outside this circle. As soon as we express and realise it our used sight of this picture will change into a geometrical sight. Under the dot signed as a letter T we see an irreducible geometrical point, under the lines t, u coming out parts of perfectly straight geometrical lines, which isn't possible to make thinner and under the line k perfect geometrical circle. Hereat the straight line t touches the circle at one point T and simultaneously this point is one of the two intersections of circles k, l etc. In short, we understand this picture as looking through it into the geometrical world.



But we won't get into the geometrical world by physical senses. However, we look into this world namely by this particular geometrical vision. Through this we look as far as the individual points, perfectly straight abscissas, which isn't possible to make thinner, etc. If we weren't able to do it, we couldn't agree for example on a fact that the straight line touches the circle at one point. We don't select axioms of euclidean geometry arbitrarily but we want the axioms to be true in this geometrical world in which we are able to look through geometrical vision. It's possible to interpret the geometrical vision as extremely culminated optic vision. Lines t, k in the mentioned picture have a common fuzz part. If we thin them, adjust t and round k, so smooth them down, their common part will reduce. After seeing through the geometrical world, where the line t becomes a real straight line and the line k becomes a real circle – their common part will reach the irreducible geometrical point. The mentioned interpretation enables us to place geometrical objects and the whole geometrical world except the horizon itself limiting possibilities of our vision by the physical sight and geometrical vision to interpret as an ability to see as far as this horizon by a certain mental sight. Every our look in any direction is limited by the horizon. Even though we can move away or bring the horizon nearer in any case we are not able to look further than as far as the horizon. The horizon such as this is an irremovable guide of our understanding to the world. The horizon, we placed the geometrical world on, has naturally a special character because it misses something which is typical for the horizon. It is namely stiff and definite which enabled that geometry took place just on it. Hereat we understand the stiffness as this horizon is still situated in the same distance from us; it's not possible to move it away.

As it was said before, we have an ability to look by the geometrical vision for example through the dot in the picture signed as a letter T as far as the geometrical point. However, we are not able to see either by our eyes or geometrical vision a bacterium, which turned up by accident in this place. This bacterium is placed beyond the horizon limiting our possibilities of optic vision, so beyond the horizon where we put the geometrical world. If we want to catch sight of the bacterium and we look in the direction of it, then we won't look further than the geometrical point. The bacterium is consequently smaller then the geometrical point; from behind the mentioned horizon it's projected into the geometrical point. But if we use a microscope we will see the bacterium. Now it is already in front of the horizon limiting our abilities of the optic vision. It's going to be so big that its body will not cover only one point but as well the whole abscissa. So from this point of view the bacterium will be bigger than the geometrical point.But also in just mentioned case the geometrical points will stay the same as we saw them before without the microscope. No geometrical point was reduced; we extended the bacterium. By means of the microscope we wanted to look inside the geometrical point because the bacterium supposed to be hidden there. But we only looked at the bacterium, not inside the point. The geometrical points ran away again into the depth where they occur all the time, no matter if we use our own eyes or the microscope. The horizon where the geometrical points are shown, isn't fixed to the real world, but it's connected with us, with our site of observation. By means of the microscope we put the bacterium in front of this horizon in such a way that thanks the alternation of the observation site we placed this horizon beyond the bacterium. We must remark that not even in a notion we will be able assume such an observation site from which the whole real world would take place in front of the horizon limiting abilities of the optic vision, so in front of the horizon on which we placed the whole geometrical world.

Even though we have the geometrical points during the geometrical vision almost within reach of hand, they àre infinitely distant from us. The passage of them is infinite in accordance with an infinite passage to the horizon. We'll call this infinite as the natural infinite. The geometrical world placed on the horizon limiting our abilities of optic vision – this is the geometrical world in which we look by geometrical vision – we'll call it as the natural geometrical world. We must remind that the euclidean geometry is obtained in this world.

The mathematics of the modern times doesn't consider the natural geometrical world as an object of geometrical studies, but it also doesn't want to know this world. The geometrical point is still always without compare smaller than the bacterium and it's not possible to place inside it an atom, the less so the bacterium. Such concepts as ,,a horizon" or ,,an observation site" are subjective and they don't belong to the european science of modern times, which is based on a fact that it's objective. The object of geometrical studies of modern times is a classical geometrical world. In this classical geometrical world the points, abscissas, circles and other geometrical objects sank down into absolutely infinite depth beyond the horizon limiting our abilities of the optic vision. In this classical geometrical world the passage to the geometrical point is absolutely infinitely long as well as the continuos lengthening of the abscissas for the given length or reaching absolutely accurate boundary of any geometrical object etc. Although we can't look inside the classical geometrical world, we still have a perfect science about it: geometry. That is so because we consider a knowledge of the natural geometrical world as a knowledge of the classical geometrical world. That's why for example the axioms of euclidean geometry are obtained in the classical geometrical world etc. To this effect are both of these worlds identical. They are inherently indistinguishable, very statement, which is obtained in one of them, is also obtained in the second one.

The existence of the classical geometrical world is more speculative than the existence of the natural world. More detailed phenomenological analysis of what the classical geometrical world is indeed, we can read in a dissertation by professor Vopěnka of thesis included in *První rozpravy s geometrií* or in his books about an alternative set theory.

Not only geometrical objects belong to the geometrical world, but also everything shown of them. Together with abscissas there also belong their length as well as real numbers and consequently various relations among them, also the real functions, sequences of numbers or functions etc. To the geometrical world we'll include everything that is somehow possible to model on it. We'll do so because in the calculus infinitesimalis the calculations take place above all in a number domain while during a selection and justification of some of their rules we are often led by a geometrical opinion.

But we have two euclidean geometrical worlds: natural and classical. So as we could express briefly, we'll label the first of them as a letter $\mathcal{N}$ and the second one as a letter $\mathcal{C}$. If we undertake just in one of them then it's indifferent which of them it is. The knowledge acquired in one them refer in such case to the second one too. But we can study them both at the same time, namely in the relationship they are located. Then in the relationship of expansion $\mathcal{N}$ to $\mathcal{C}$ because the classical geometrical world arised by spreading of the natural geometrical world into the classical infinite. This inner undistinguishability of both mentioned geometrical worlds we can take down by the subsequent method:

The first law of the expansion. *If ψ is a quality of the objects belonging to $\mathcal{N}$, which is formulated only by means of the world $\mathcal{N}$* (consequently without contemplating the existence of the world $\mathcal{C}$), *then ψ is also a quality of the objects belonging to $\mathcal{C}$* (consequently without contemplating the existence of the world $\mathcal{N}$). *Nevertheless in the world $\mathcal{N}$ there exists an object having a quality ψ iff an object having a quality ψ also exists in the world $\mathcal{C}$.*

From here follows:

All the objects of the world $\mathcal{N}$ have a quality ψ iff all the objects of the world $\mathcal{C}$ have a quality ψ.

As soon as we also include numbers in the geometrical world, then it is true that these numbers like $1, 2, 3, \ldots$ belong to $\mathcal{N}$ as well as $\mathcal{C}$. This refers to all real numbers which belong to the world $\mathcal{N}$. Identification of points of the geometrical world with the ordered n-couples of real numbers allows to explicate not only real numbers this way, but all the objects belonging to $\mathcal{N}$ at all. Consequently we accept following law which takes down this presumptive identity of the worlds $\mathcal{N}$ and $\mathcal{C}$.

The second law of the expansion. *The objects belong to $\mathcal{N}$ also belong to $\mathcal{C}$* (during the expansion of $\mathcal{N}$ they came into $\mathcal{C}$) *and have the same qualities in the world $\mathcal{C}$ and they are placed in it to the same relationship* (formalized only by the means of the world $\mathcal{N}$ and namely just by the means of the world $\mathcal{C}$) *as in the world $\mathcal{N}$.*

From here follows next helpful rule.

The rule of defined objects. *Let ψ is the quality of the objects, which formalized only by means of the world $\mathcal{N}$ (and namely just by means of the world $\mathcal{C}$). Let just one object A exists in the world $\mathcal{C}$, which has the quality ψ. Then the object A belongs to the world $\mathcal{N}$ and is the only one object of the world $\mathcal{N}$, which has the quality ψ.*

If $a_1, a_2, \ldots, a_n$ is sequence of the objects belonging to the world $\mathcal{N}$, which is finite in this world. According to the second law of the expansion this sequence belongs, as his each member, to the world $\mathcal{C}$ too. Nevertheless the amount of members won't increase in the world $\mathcal{C}$. But it's not the same with infinite sequences. Nevertheless the case of sequence **N** of all positive integer is the key case. Even this sequence belongs to $\mathcal{N}$ as well as to $\mathcal{C}$ and both of these worlds has the same qualities. Its each member belonging to $\mathcal{N}$ belongs understandably to $\mathcal{C}$ too. But there will increase the amount of new member in the world $\mathcal{C}$ to **N** because in $\mathcal{N}$ is this sequence naturally infinite while in $\mathcal{C}$ it's infinite classically and so here it's much longer. Nevertheless according to the preceding thinking no new positive integer will be added in front of any number belonging to $\mathcal{N}$. This is expressed by the following law, which is the last of the laws of the expansion.

The third law of the expansion. *In the expansion of $\mathcal{N}$ to $\mathcal{C}$ the* **N** *sequence of all positive integer will extended for new positive integer belonging to $\mathcal{C}$ and not to $\mathcal{N}$.*

Therefore during the expansion $\mathcal{N}$ to $\mathcal{C}$ in like manner will also extend every infinite (taking it as natural infinite) sequence belonging to $\mathcal{N}$.

These positive integer in the world $\mathcal{C}$, which belong to $\mathcal{N}$, are called finite big numbers. System of these numbers we'll sign as **FN**.

These positive integer in the world $\mathcal{C}$, which accrued to **N** *after the expansion of $\mathcal{N}$ to $\mathcal{C}$, are called infinite big numbers. System of these numbers we'll sign as* **IN**.

We can't separate the systems **FN**, **IN** from the **N** by interior means of the world $\mathcal{C}$. Actually these systems apprehended separately don't belong to the world $\mathcal{C}$. They came there outwardly. It's possible to separate them from **N** as lately as we have a chance to compare both of the geometrical worlds $\mathcal{N}$, $\mathcal{C}$ in the relation of expansion $\mathcal{N}$ to $\mathcal{C}$.

If $n \in \mathbf{FN}$, $m < n$, *then* $m \in \mathbf{FN}$. *If* $m \in \mathbf{IN}$, $m < n$, *then* $n \in \mathbf{IN}$.

In the world $\mathcal{N}$ is the sequence **N** composed by only those positive integer, which are the elements of the **FN**. Owing to this in the world $\mathcal{C}$ obtains:

If $m, n \in \mathbf{FN}$, *then* $m + n \in \mathbf{FN}$, $nm \in \mathbf{FN}$, $m^n \in \mathbf{FN}$.

Although in the world $\mathcal{C}$ is **IN** *non-empty part of the sequence of all positive integer, it still doesn't have the smallest element.*

The real number x belonging to $\mathcal{C}$ is infinite big, iff $\forall n \in \mathbf{FN}$ *is* $n < |x|$.

The real number x belonging to $\mathcal{C}$ is infinite big, iff $\exists n \in \mathbf{IN}; n < |x|$.

The real number belonging to $\mathcal{C}$ is finite big, iff it's not infinite big.

The real number x belonging to $\mathcal{C}$ is finite big, iff $\forall n \in \mathbf{IN}$ *is* $|x| < n$.

The real number x belonging to $\mathcal{C}$ is finite big, iff $\exists n \in \mathbf{FN}$ *is* $|x| < n$.

The real number x belonging to $\mathcal{C}$ is infinite small, iff $x = 0$ *or* $x \neq 0$ *and*

$\frac{1}{x}$ is infinite big.
The real number x belonging to $\mathcal{C}$ is infinite small, iff $\forall n \in \mathbf{FN}, n \neq 0$ is $|x| < \frac{1}{n}$.
The real number x belonging to $\mathcal{C}$ is infinite small, iff $\exists n \in \mathbf{IN}$ and $|x| < \frac{1}{n}$.
Every real number, which belong to $\mathcal{N}$ is finite big.
No real number, which is not equal by 0, belonging to $\mathcal{N}$ isn't infinite small.
We will say, that the real numbers x, y belonging to $\mathcal{C}$ are infinite proximal $x \doteq y$, iff $x - y$ is infinite small.
If x, y, z are real numbers belonging to $\mathcal{C}$. Then evidently:
(a) $x \doteq x$.
(b) *If $x \doteq y$, then $y \doteq x$.*
(c) *If $x \doteq y$, $y \doteq z$, then $x \doteq z$*
(d) *If $x \doteq y$, then if x is infinite big, then y is also infinite big; if x finite big, then y is also finite big; if x is infinite small, then y is also infinite small.*
If x, y, u, v are real numbers belonging to $\mathcal{C}$ such that $x \doteq y$ a $u \doteq v$. Then:
(a) $x + u \doteq y + v, \quad x - u \doteq y - v$.
(b) *If x, u are finite big, then $xu \doteq yv$.*
(c) *If x isn't infinite small, then $\frac{1}{x} \doteq \frac{1}{y}$.*
(d) *If x, u are finite big and x isn't infinite small, then $\frac{u}{x} \doteq \frac{v}{y}$.*
If x, y are the numbers belonging to $\mathcal{N}$, then $x \doteq y$, iff $x = y$.

If we try to look as far as some geometrical point X situated in the geometrical world $\mathcal{C}$, which distance from our observation site is a finite big real number, then we don't to this point. Our vision will stop at some point Y situated in the world $\mathcal{N}$ whereby the point X is covered. Videlicet, the point X is by return projected to the point Y. After the expansion of $\mathcal{N}$ to $\mathcal{C}$ when the point Y falls in the world $\mathcal{C}$, the points X, Y are in this world infinite proximal. If we put this thought into a language of real numbers, we'll get the following law.

The first law of the reverse projection. *During the reverse projection of $\mathcal{C}$ on $\mathcal{N}$ every finite big real number x belonging to $\mathcal{C}$ is projected to its infinite proximal real number y belonging to $\mathcal{N}$.*

If x is finite big real number belonging to $\mathcal{C}$, then exists one and only one real number belonging to $\mathcal{N}$, which we'll sign as $Proj(x)$, such that $x \doteq Proj(x)$.

If x, y are finite big real numbers belonging to $\mathcal{C}$. Then is:
(a) $Proj(x + y) = Proj(x) + Proj(y)$.
(b) $Proj(x - y) = Proj(x) - Proj(y)$.
(c) $Proj(xy) = Proj(x) \cdot Proj(y)$.
(d) *If $Proj(y) \neq 0$, then is $Proj(\frac{x}{y}) = \frac{Proj(x)}{Proj(y)}$.*

If x, y are finite big real number belonging to C, $x < y$. Then is $Proj(x) \leq Proj(y)$, and $Proj(x) = Proj(y)$, iff $x \doteq y$.

If we try to look as far as some infinite big real number belonging to C, our vision will stop at some vanishing point which additionally appeared behind all the real numbers belonging to $\mathcal{N}$. This leads us to the acceptance of the following law.

The second law of the reverse projection. *During reverse projection C to $\mathcal{N}$ every positive (negative) infinite big real number belonging to C is projected to the additional appeared number in the world $\mathcal{N}$, which is signed ∞ ($-\infty$).*

The world $\mathcal{N}$ is consequently extended for numbers ∞, $-\infty$. Originally these numbers didn't belong there: they appeared there just after the expansion $\mathcal{N}$ to C. They are consequently *improper real numbers* belonging to $\mathcal{N}$. We can now extend the operation of the reverse projection to all real numbers belonging to C, namely by the following way:
Let x is infinite big real number belonging to C. If $x > 0$, then $Proj(x) = \infty$; if $x < 0$, then $Proj(x) = -\infty$.

References

[1] Vopěnka, P.: *Calculus Infinitesimalis*, Práh, Praha, 1997

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