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RANDOMLY kC_n GRAPHS

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Abstract. A graph G is said to be a randomly H graph if and only if any subgraph of G without isolated vertices, which is isomorphic to a subgraph of H, can be extended to a subgraph H_1 of G such that H_1 is isomorphic to H. The problem of characterization of randomly H graphs, where H is $r-$regular graph on p vertices, was given by Tomasta and Tomová. In general, the characterization of such graphs seems to be difficult. However, there exist several results for the case $r = 2$. Chartrand, Oellermann, and Ruiz characterized randomly C_n graphs. Híc and Pokorný characterized randomly $2C_n$ graphs, as well as randomly $C_n \cup C_m$ graphs, where $n \neq m$. In this paper, the problem of randomly H graphs, where $H = kC_n, k > 2$ is discussed.

1. Introduction

In 1986 Chartrand, Oellermann, and Ruiz [2] introduced the term 'randomly H graph' as follows: Let G be a graph containing a subgraph H without isolated vertices. Then G is called a randomly H graph if whenever F is a subgraph of G without isolated vertices that is isomorphic to a subgraph of H, then F can be extended to a subgraph H_1 of G such that H_1 is isomorphic to H.

Note that a subgraph H_1 does not have to maintain the partial isomorphism between F and the subgraph of H.

In [1] and [2] authors pointed out the problems concerning isolated vertices in the definition of randomly H graphs. That is why we consider that both H and F are free of isolated vertices.

Every nonempty graph is randomly K_2, while every graph G without isolated vertices is a randomly G graph. K_n is randomly H for every

$H \subseteq K_n$. The graph $K_{3,3}$ is randomly H for every subgraph H of $K_{3,3}$ (see [2, Theorem 1]).

2. Preliminaries

The general question is 'For what classes of graphs H is it possible to characterize all those graphs G that are randomly H?'.

In [5] the characterization of randomly $K_{r,s}$ graphs was given, but in terms of H-closed graphs. In [1] Alavi, Lick, and Tian studied randomly complete n-partite graphs and characterized them.

The problem of characterization of randomly H graphs, where H is r-regular graph on p vertices, was given by Tomasta and Tomová (see [9]). In general the characterization of such graphs seems to be difficult. However, there exist several results for $r = 2$. The list of the most important results about randomly $2-$regular graphs follows.

Theorem A (see Chartrand, Oellermann, and Ruiz [2]) A graph G is randomly C_3 if and only if each component of G is a complete graph of order at least 3.

Theorem B (see Chartrand, Oellermann and Ruiz [2]) A graph G is randomly C_4 if and only if

1. $G = K_p$, where $p \geq 4$, or
2. $G = K_{r,s}$, where $2 \leq r \leq s$.

Theorem C (see Chartrand, Oellermann, and Ruiz [2]) A graph G is randomly C_n, $n \geq 5$, if and only if

1. $G = K_p$, where $p \geq n$, or
2. $G = C_n$, or
3. $G = K_{n/2,n/2}$ and n is even.

Theorem D (see Tomasta and Tomová [9]) Let H be connected n-vertex regular graph of degree $r \geq 2$ different from K_3 and C_4. Then a p-vertex graph G where $p > n$ is randomly H if and only if $G = K_p$.

Theorem E (see Híc and Pokorný [7]) A graph G is randomly $2C_3$ if and only if

1. $G = K_p,\ p \geq 6$, or
2. $G = K_{p_1} \cup K_{p_2} \cup ... \cup K_{p_n}$, where $n \geq 2$, $p_i = 3$ or $p_i \geq 6$.

Theorem F (see Híc and Pokorný [7]) A graph G is randomly $2C_{2n+1}$, where $n \geq 2$, if and only if

1. $G = 2C_{2n+1}$, or
2. $G = 2K_{2n+1}$, or

3. $G = C_{2n+1} \cup K_{2n+1}$, or
4. $G = K_p$, $p \geq 2(2n+1)$.

Theorem G (see Híc and Pokorný [7]) A graph G is randomly $2C_4$ if and only if
1. $G = K_{r,s}$, where $4 \leq r \leq s$, or
2. $G = 2C_4$, or
3. $G = 2K_4$, or
4. $G = C_4 \cup K_4$, or
5. $G = K_p$, where $p \geq 8$.

Theorem H (see Híc and Pokorný [7]) A graph G is randomly $2C_{2n}$, where $n \geq 3$, if and only if
1. $G = 2K_{2n}$, or
2. $G = 2C_{2n}$, or
3. $G = 2K_{n,n}$, or
4. $G = C_{2n} \cup K_{n,n}$, or
5. $G = C_{2n} \cup K_{2n}$, or
6. $G = K_{n,n} \cup K_{2n}$, or
7. $G = K_{2n,2n}$, or
8. $G = K_p$, $p \geq 4n$.

Theorem I (see Híc and Pokorný [8]) A graph G is randomly $C_n \cup C_m$, where $3 \leq n < m$ if and only if
1. $G = C_n \cup C_m$, or
2. $G = K_n \cup C_m$, or
3. $G = K_{\frac{n}{2},\frac{n}{2}} \cup C_m$ where n is even, or
4. $G = K_{\frac{m+n}{2},\frac{m+n}{2}}$ where both m and n are even, or
5. $G = K_p$, where $p \geq m+n$.

This paper deals with randomly 2− regular graphs H, where $H = kC_n$ for $k \geq 3$. All the terms used in this paper can be found in [4]. Especially, if H is a subgraph of G, we will use $G - H = \langle V(G) - V(H) \rangle$ to denote the induced subgraph of the graph G with the vertex set $V(G) - V(H)$.

3. Results

Theorem 1 A graph G is randomly kC_3 if and only if
(i) $G = K_p,\ p \geq 3k$, or
(ii) $G = K_{p_1} \cup K_{p_2} \cup ... \cup K_{p_n}$, where $n \geq 2$, $p_i = 3$ or $p_i \geq 3k, \sum p_i \geq 3k$.

Proof: The sufficiency is obvious, as both $G = K_p$, where $p \geq 3k$, and $G = K_{p_1} \cup K_{p_2} \cup ... \cup K_{p_n}$, where $n \geq 2$, $p_i = 3$ or $p_i \geq 3k$, are randomly kC_3 graphs.

For necessity, let G be a randomly kC_3 graph. Let G be connected. We will prove that no edge is missing.

Assume the contrary. Let the edge $\{u, v\}$ be missing. Since G is connected, there exists a $u - v$ path. Let us choose the shortest $u - v$ path P. Obviously, the length of P is at least 2. Then no subpath of P of the length 2 cannot be expanded into C_3, a contradiction with the assumption that G is a randomly kC_3 graph.

Now, let G not to be connected. Then G cannot contain components K_{p_i}, where $p_i \neq 3, p_i < 3k$. The case $p_i = 2$ is trivial. If $p_i \in 4, 5, 6, ..., k-1$ those components contain a subgraph F isomorphic to $\lfloor \frac{p_i}{2} \rfloor K_2$. Although F is isomorphic to a subgraph of kC_3, F cannot be extended to kC_3, a contradiction. If $p_i \in k, k+1, k+2, ..., 3k-1$ those components contain a subgraph F isomorphic to kK_2. Although F is isomorphic to a subgraph of kC_3, F cannot be extended to kC_3, a contradiction. If some component of G has exactly three vertices, then the component is isomorphic to K_3, as G is a randomly kC_3 graph. If some component of G has more than $3k - 1$ vertices, then the component is a complete subgraph of G. Otherwise, we can produce a contradiction similarly to the connected case.

Lemma 1: Let G be a disconnected randomly kC_n graph, where $n > 3$. Then G has k components and each of the components has exactly n vertices.

Proof: a) Let G have more than k components. Let a subgraph F of G consist of $k+1$ edges which belong to $k+1$ different components of G. Subgraph F is isomorphic to some subgraph of kC_n. However, the subgraph F cannot be extended to kC_n, a contradiction.

b) Let G have less than k components. Then there exists a component that has more than n vertices. Let C be a component of G which has at least $n+1$ vertices. Then there exists a subgraph F_1 of C, which consists of a path on $n-1$ vertices and of one edge. Let $\{u, v\}$ be an edge of the other component of G. Let $F = F_1 \cup \{u, v\}$. Obviously F is a subgraph of G which is isomorphic to a subgraph of kC_n. However, F cannot be extended to kC_n, a contradiction.

c) Let G have k components. Obviously none of the components has less than n vertices. Similarly to case b we can prove that none of the components has more than n vertices.

According to a), b) and c), G has k components and each component has n vertices.

Theorem 2: A graph G is randomly kC_{2n+1}, where $n \geq 2$, if and only if
(i) $G = H_1 \cup H_2 \cup ... \cup H_k$, where H_i is either C_{2n+1} or K_{2n+1}, or
(ii) $G = K_p$, $p \geq k(2n+1)$.

Proof: It is not difficult to verify that each of the graphs listed in the statement of the theorem has the desired property. For the converse we assume that G is a randomly kC_{2n+1} graph.

1. Assume first that G is disconnected. We will prove that G fits part (i) in the statement of the theorem.

Lemma 1 implies that G has k components and each component has exactly $2n+1$ vertices. Since G is randomly kC_{2n+1}, each component is randomly C_{2n+1}. According to Theorem C, each component of G is either $G = C_{2n+1}$ or $G = K_{2n+1}$, so G fits part (i).

2. We henceforth assume that G is connected. We will prove that $G = K_p$, $p \geq k(2n+1)$. Let us use the mathematical induction. If $k = 2$, then by Theorem F, part 4 holds $G = K_p$. Assume that the statement holds for every $r < k$. Let H be a subgraph of G which is isomorphic to C_{2n+1}.

a) Let $G' = G - H$. Let us notice that G' is a randomly $(k-1)C_{2n+1}$ graph. By the inductive hypothesis, G' is a complete graph. Now we will prove that $G'' = \langle V(H) \rangle$ is complete, too. Assume the contrary.

Let $G'' = C_{2n+1}$. Then $V(G'') = V(H) = \{v_1, v_2, ..., v_{2n+1}\}$ and $E(G'') = \{\{v_i, v_{i+1}\}; i = 1, 2, ...2n\} \cup \{\{v_{2n+1}, v_1\}\}$. Since G is connected, there exists an edge $\{u, v\}$, where $u \in V(H)$, $v \in V(G')$. Without loss of generality we may assume that $v = v_1$. Let us construct the path $u, v_1, v_2, ..., v_{2n-1}, v_{2n}$. This path can be extended to C_{2n+1} only by adding the edge $\{v_{2n}, u\}$. Now let us construct the path $v_{2n+1}, v_{2n}, u, v_1, v_2, ..., v_{2n-3}, v_{2n-2}$. This path can be extended to C_{2n+1} only by adding $\{v_{2n-2}, v_{2n+1}\}$. So G'' is not isomorphic to C_{2n+1}, a contradiction. Then, by Theorem C, $G'' = K_{p'}$, $p' = 2n+1$.

b) Now it is necessary to prove that for every $u \in V(G')$, $v \in V(G'')$ the graph G contains the edge $\{u, v\}$. It is sufficient to choose $u-v$ path on $2n+1$ vertices. Since both G' and G'' are complete and G is connected, the path always exists and can be extended to C_{2n+1} only if $\{u, v\}$ edge exists. Since both u and v are arbitrary vertices, G is complete.

The formula $p \geq k(2n+1)$ follows from the fact that G is randomly kC_{2n+1}.

Lemma 2: Let G be a randomly kC_4 graph, $K_{r,s} \subset G$, $K_{r,s} \neq G$, where $2k \leq r \leq s$. Let $V(G) = V(K_{r,s})$. Then G is a complete graph.

Proof: Let $G \supset K_{r,s}$ and $V(G) = V(K_{r,s}) = \{u_1, ..., u_r\} \cup \{v_1, ..., v_s\}$. Let $\{u_i, u_j\} \in E(G)$, $\{u_i, u_j\} \notin E(K_{r,s})$. Let v_q and v_l be arbitrary vertices which belong to the different partite set than u_i and u_j. Let us construct the path v_q, u_i, u_j, v_l. Because G is randomly kC_4, the path can be extended to C_4 only by adding the edge $\{v_q, v_l\}$. Since v_q and v_l are arbitrary vertices, $\{v_q, v_l\} \in E(G)$ for every q, l. If we use a similar method with the edge $\{v_i, v_j\} \in E(G)$, we prove that G is a complete graph.

Theorem 3: A graph G is randomly kC_4 if and only if
(i) $G = K_{r,s}$, where $2k \leq r \leq s$, or
(ii) $G = H_1 \cup H_2 \cup ... \cup H_k$, where H_i is either C_4 or K_4 , or
(iii) $G = K_p$, where $p \geq 4k$.

Proof: It is not difficult to verify that each of the graphs listed in the statement of the theorem has the desired property. For the converse then, we assume that G is randomly kC_4.

1. Assume first that G is disconnected. We will prove that G fits part (ii) in the statement of the theorem.

Lemma 1 implies that G has k components and each component has exactly four vertices. Since G is randomly kC_4, each of its component has to be randomly C_4. According to Theorem B the graph G fits the part (ii) in the statement of the theorem.

2. We henceforth assume that G is a connected randomly kC_4 graph. We will prove that either $G = K_{r,s}$, where $2k \leq r \leq s$ or $G = K_p$, where $p \geq 4k$. We will use the mathematical induction. If $k = 2$, then it is true by Theorem G, parts 1 and 5.

Assume that the statement holds for every $r < k$. Let H be a subgraph of G isomorphic to C_4. Let $G' = G - H$. Since G is randomly kC_4, G' is randomly $(k-1)C_4$. By the inductive hypothesis, either $G' = K_t$, $t \geq 4(k-1)$ or $G' = K_{a,b}$, $2(k-1) \leq a \leq b$.

a) Let $G' = K_{a,b}$, $2(k-1) \leq a \leq b$. Let $\{w_1, w_2, ..., w_b\}$ and $\{s_1, s_2, ..., s_a\}$ be a partition of $V(G')$ into two sets. Let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be a partition of $V(H)$ into two sets. Since G is connected, it contains an edge which starts in $V(H)$ and ends in $V(G')$. Without loss of generality we may assume that the edge is $\{v_1, w_1\}$. Since the graph G is randomly kC_4, every path u_i, v_1, w_1, s_j can be extended to C_4 by adding the edge $\{s_j, u_i\}$ for every $i = 1, 2$ and $j = 1, 2, ..., a$. Using the same method with the edge $\{u_1, s_1\}$ we prove that the graph G contains all edges $\{v_i, w_j\}$ for every $i = 1, 2$ and $j = 1, 2, ..., b$. So $G \supset K_{a+2,b+2}$ and $V(G) = V(K_{a+2,b+2})$.

If $G \neq K_{a+2,b+2}$, by Lemma 2 $G = K_p$, $p \geq 4k$. Otherwise $G = K_{r,s}$, where $2k \leq r \leq s$.

b) Let $G' = K_t$, $t \geq 4$. Using the same method as in part a) it is not difficult to prove that G is a complete graph, which means that $G = K_p$, $p \geq 4k$.

Lemma 3: Let G be a randomly kC_{2n} graph, $K_{2n,2n} \subset G$, where $n > 2$. Let $V(G) = V(K_{2n,2n})$. Then G is a complete graph.

Proof: The proof is similar to that of Lemma 2. A major difference is that we have to consider a path on $2n$ vertices instead of a path on 4 vertices.

Theorem 4: A graph G is randomly kC_{2n}, where $n \geq 3$, if and only if
(i) $G = H_1 \cup H_2 \cup ... \cup H_k$, where H_i is C_{2n} or K_{2n} or $G = K_{n,n}$, or
(ii) $G = K_{kn,kn}$, or
(iii) $G = K_p$, $p \geq 2kn$.

Proof: It is not difficult to verify that each of the graphs listed in the statement of the theorem has the desired property. For the converse then, we assume that G is randomly kC_{2n}.

1. Assume first that G is disconnected. We will prove that G fits part (i) in the statement of the theorem.

Lemma 1 implies that G has k components and each component has exactly $2n$ vertices. Since G is randomly kC_{2n}, each of the components is randomly C_{2n}.

According to Theorem H, parts 1-6, G fits the part (i).

2. We henceforth assume that G is connected.

a) Let $|V(G)| > 2kn$. We will prove that G is complete. We will use the mathematical induction. If $k = 2$, it is true by Theorem H, part 8.

Assume that the statement holds for every $r < k$. Let H be a subgraph of G isomorphic to C_{2n}. Let $G' = G - H$. Since G is randomly kC_{2n}, G' is a randomly $(k-1)C_{2n}$ graph of order $p' > 2(k-1)n$. Therefore, by the inductive hypothesis, G' is complete.

Similarly, if we choose a subgraph of G' isomorphic to C_{2n}, we will obtain that $\langle V(H) \rangle$ is complete.

Now it is sufficient to prove that for every $v \in V(H)$ and for every $w \in V(G')$ there exists the edge $\{v, w\} \in E(G)$. As G is connected and both $\langle V(H) \rangle$ and G' are complete, there exists a $v - w$ path on $2n$ vertices. Let us choose this path. Because G is randomly kC_{2n}, the path can be extended to C_{2n} by adding the edge $\{v, w\}$. That is why G is complete.

b) Let $|V(G)| = 2kn$. Using mathematical induction we will prove that either $G = K_{kn,kn}$ or $G = K_p, p \geq 2kn$. If $k = 2$, it is true by Theorem H, cases 7 and 8.

Assume that the statement holds for every $r < k$. Let us construct graphs H and G' similarly to case a). Since the order of G' is $2(k-1)n$ and G is randomly kC_{2n}, by the inductive hypothesis, G' is isomorphic to $K_{(k-1)n,(k-1)n}$ or $K_{p'}, p' = 2(k-1)n$.

b$_1$) If G' is isomorphic to $K_{p'}, p' = 2(k-1)n$ or $K_{(k-1)n,(k-1)n}$. Similarly to Theorem 2, part a we can prove that $\langle V(H) \rangle$ is not isomorphic to C_{2n}.

b$_2$) Let G' be isomorphic to $K_{(k-1)n,(k-1)n}$ and $\langle V(H) \rangle$ be isomorphic to $K_{n,n}$. Let $\{w_1, w_2, ..., w_{(k-1)n}\}$ and $\{s_1, s_2, ..., s_{(k-1)n}\}$ be a partition of $V(G')$ into two sets. Let $\{u_1, u_2, ..., u_n\}$ and $\{v_1, v_2, ..., v_n\}$ be a partition of $V(H)$ into two sets. We will prove that for every $i, j \in \{1, 2, ..., n\}$ $E(G)$ contains the edge $\{s_i, v_j\}$. Since G is connected, it contains an edge which starts in $V(H)$ and ends in $V(G')$. Without loss of generality we may assume that the edge is $\{v_1, w_1\}$. Let us construct the path $v_i, u_1, w_1, s_1, w_2, s_2, ..., w_{j-1}, s_{j-1}, w_{j+1}, s_{j+1}, ..., s_{n-1}, w_n, s_j$. Since G is randomly kC_{2n}, the path can be extended to C_{2n} by adding the edge $\{s_j, v_i\}$, so $\{s_j, v_i\} \in E(G)$.

Using the similar method we can prove that $\{w_i, u_j\} \in E(G)$ for every $i, j \in \{1, 2, ..., n\}$. So $G \supset K_{kn,kn}$. If $G \neq K_{kn,kn}$, then by Lemma 3 $G = K_{2kn}$.

b$_3$) If G' is isomorphic to $K_{2(k-1)n}$ and $\langle V(H) \rangle$ is isomorphic to K_{2n} or $K_{n,n}$, then similarly to b$_2$ case we can prove that $G \supset K_{kn,kn}$ and $G \neq K_{kn,kn}$. Then, by Lemma 3, $G = K_{2kn}$.

4. Conclusion

In the paper a complete characterization of randomly H graphs where $H = kC_n$ is given. The case of 2−regular randomly H graphs, which contain more than two circuits of which at least two are different, remains open.

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