Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie

Matematyka IX, Częstochowa 2003

Several Remarks about Three-valued Kleene's Propositional Logic, without Tautologies

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In this article we give, in the syntetic way, diferent formal aproachings to Kleene's propositional logic. In the work [5] S. C. Kleene gives a three-valued sentential calculus characterized by the following matrix:

$$\mathfrak{M}_K = (\{0, 1/2, 1\}, \{1\}, \{\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow\}).$$

The operators of the matrix are given by the formulas:

$\neg x = 1 - x, \quad x \vee y = \max(x, y), \quad x \wedge y = \min(x, y)$

$x \Rightarrow y = \neg x \vee y, \quad x \Leftrightarrow y = (x \Rightarrow y) \wedge (y \Rightarrow x).$

The operators have the tables:

$\vee$	0	$\frac{1}{2}$	1	$\neg$
0	0	$\frac{1}{2}$	1	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
1	1	1	1	0

$\wedge$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\Rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	0	$\frac{1}{2}$	1

$\Leftrightarrow$	0	$\frac{1}{2}$	1
0	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

It is easy to see that the set of tautologies is an empty set.

The matrical consequence adequate for the matrix $\mathfrak{M} = (\mathrm{U}, \mathrm{V}, \mathrm{F})$ we define as follows:

Definition 1. $\alpha \in C_{\mathfrak{M}}(X) \Leftrightarrow \forall_{h \in Hom(S,U)} (h(X) \subseteq V \Rightarrow h(\alpha) \in V).$

S is the set of all wellformed expressions, which have been created from sentential variables and the operators, V is the set of designated elements of U, F is the set of operators of algebra (U, F). The symbol Hom(S, U) will be used to denote the set of all homomorphisms of the language S into the algebra (U, F).

Definition 2. $E(\mathfrak{M}) = C_{\mathfrak{M}}(\emptyset)$.

From the definitions 1 and 2 we have the following conclusion:

Conclusion 1. $E(\mathfrak{M}) = \{\alpha \in S : \forall_{h \in Hom(S,U)} (h(\alpha) \in V)\}$.

To proof that $E(\mathfrak{M}_K) = \emptyset$ for any expresion $\alpha = \alpha(p_1, p_2, ..., p_n)$ we take a mapping h such that: $h(p_1) = h(p_2) = ... = h(p_n) = 1/2$. We have now $h(\alpha) = 1/2 \notin \{1\}$. The matrix $\mathfrak{M}_K$ is not the same like the three-valued matrix of Łukasiewicz: $\mathfrak{M}_L = (\{0, 1/2, 1\}, \{1\}, \{\neg, \vee, \wedge, \rightarrow_L, \leftrightarrow_L\})$, because the operators $\rightarrow_L, \leftrightarrow_L$ have the tables:

$\rightarrow_L$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

$\leftrightarrow_L$	0	$\frac{1}{2}$	1
0	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

For $x = y = 1/2$ we have $x \rightarrow_L y = 1, x \leftrightarrow_L y = 1, x \Rightarrow y = 1/2$, $x \Leftrightarrow y = 1/2$. The operators $\neg, \vee, \wedge$ are the same like the operators of the matrix $\mathfrak{M}_K$. For the matrix $\mathfrak{M}_L$ the set $E(\mathfrak{M}_L)$ is not an empty set.

The matrix $\mathfrak{M}_K$ is isomorphic to the following matrix given by Simons in the work [6]:

$\mathfrak{M}_C = (\{0, 1, 2\}, \{0\}, \{\sim, +, \circ, \rightarrow, \leftrightarrow\})$.

The operators are given by the formulas: [1]
$\sim x = 2 - x, \quad x + y = \min(x, y), \quad x \circ y = \sim(\sim x + \sim y) = \max(x, y)$,
$x \rightarrow y = \sim x + y, \quad x \leftrightarrow y = (x \rightarrow y) \circ (y \rightarrow x)$.

The tables of those operators are following:

+	0	1	2	$\sim$
0	0	0	0	2
1	0	1	1	1
2	0	1	2	0

$\circ$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

[1] In the oryginal version of Simon's system somewhat diferent symbols are used for some operators of the Simon's matrix.

$\rightarrow$	0	1	2
0	0	1	2
1	0	1	1
2	0	0	0

$\leftrightarrow$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	0

Theorem 1. The matrix $\mathfrak{M}_C$ and the matrix $\mathfrak{M}_K$ are isomorphic.

Proof. We define a mapping Φ as follows: the operators $\sim, +, \circ, \rightarrow, \leftrightarrow$ from $\mathfrak{M}_C$ we map into the operators $\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow$ from the matrix $\mathfrak{M}_K$, $\Phi : \{0,1,2\} \rightarrow \{0,1/2,1\} : \Phi(0) = 1, \Phi(1) = 1/2, \Phi(2) = 0$. It is easy to see that the mapping Φ is an isomorphism. Φ satisfies two conditions: (1) Φ is a bijection, (2) Φ is a homomorphism, it means: $\Phi(\sim x) = \neg\Phi(x), \Phi(x+y) = \Phi(x) \vee \Phi(y), \Phi(x \circ y) = \Phi(x) \wedge \Phi(y), \Phi(x \rightarrow y) = \Phi(x) \Rightarrow \Phi(y), \Phi(x \leftrightarrow y) = \Phi(x) \Leftrightarrow \Phi(y)$.

In his work [6] L. Simons gives a formalization of the matrix $\mathfrak{M}_C$ by the set of rules. The set of rules of inference is the following:[2]

$$r_1 : \frac{K\alpha\beta}{\alpha}, \quad r_2 : \frac{\alpha, \beta}{K\alpha\beta}, \quad r_3 : \frac{\alpha}{A\alpha\beta}.$$

The set of rules of replacement[3] is the following:

$$NK\alpha\beta \doteq AN\alpha N\beta, \quad A\alpha\beta \doteq A\beta\alpha, \quad A\alpha A\beta\gamma \doteq AA\alpha\beta\gamma,$$

$$K\alpha A\beta\gamma \doteq AK\alpha\beta K\alpha\gamma, \quad \alpha \doteq NN\alpha, \quad \alpha\beta \doteq AN\alpha\beta,$$

$$E\alpha\beta \doteq KC\alpha\beta C\beta\alpha, \quad E\alpha\beta \doteq AK\alpha\beta KN\alpha N\beta,$$

$$\alpha \doteq A\alpha\alpha, \quad A\alpha K\beta N\beta \doteq \alpha.$$

We denote the set of all rules above by the R_C symbol. In the work [6] Simons proved that:

Theorem 2. $C_{\mathfrak{M}_C} = C_{R_C}$.

C_{R_C} is a consequence based on the set of rules R_C. In the proof of theorem 2 Simons uses the transformation of expressions to a normal form. The system above is an improved version of Copi's system (see [6]).

[2] He uses operators of language: N - negation, A - disjunction, K - conjunction, C - implication, E - equivalence.

[3] Rules of replacement make it possible to replace expressions or fragments of expressions by equivalent expressions in the sense of relation $\doteq$.