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On Construction of the First-Order Differential Equations in the Space M(1,3) × R(u) with Non-trivial Symmetry

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Abstract. *We have constructed the first-order differential equations invariant under non-splitting subgroups of the group* $P(1,4)$ *and defined in the space* $M(1,3) \times R(u)$. *The results obtained can be used in relativistic and non-relativistic physics.*

It is well known, that the differential equations with non-trivial symmetry groups are wide applicable in theoretical and mathematical physics (see, for example, [1–3]).

The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group has many applications in theoretical and mathematical physics (see, for example, [3]). Non-splitting subgroups of the group $P(1,4)$ have been described in [4]. The conjugation was considered under the group $P(1,4)$.

From the results obtained in [3] it follows, in particular, that the group $P(1,4)$ contains as subgroups the Poincaré group $P(1,3)$ and the extended Galilei group $\tilde{G}(1,3)$.

The paper [5] is devoted to the construction of the first-order differential equations in the space $M(1,3) \times R(u)$, which are invariant under splitting subgroups of the group $P(1,4)$. $R(u)$ is the number axis of the dependent variable u.

In the present paper we continue to study this type of equations. We concentrate our attention on the first-order differential equations (in the space $M(1,3) \times R(u)$) invariant under non-splitting subgroups of the group $P(1,4)$.

1. The Lie algebra of the group P(1,4) and its representation

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu, \nu = 0,1,2,3,4)$ and P'_{μ} $(\mu = 0,1,2,3,4)$, which satisfy the commutation relations

$$\left[P'_{\mu}, P'_{\nu}\right] = 0, \qquad \left[M'_{\mu\nu}, P'_{\sigma}\right] = g_{\mu\sigma}P'_{\nu} - g_{\nu\sigma}P'_{\mu},$$

$$\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} \equiv iM_{\mu\nu}$.

In this work we consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \quad P'_1 = -\frac{\partial}{\partial x_1}, \quad P'_2 = -\frac{\partial}{\partial x_2}, \quad P'_3 = -\frac{\partial}{\partial x_3},$$

$$P'_4 = -\frac{\partial}{\partial u}, \quad M'_{\mu\nu} = -\left(x_{\mu}P'_{\nu} - x_{\nu}P'_{\mu}\right), \quad x_4 \equiv u.$$

About this representation see, for example, [2].

Below, we will use the following basis elements:

$$G = M'_{40}, \quad L_1 = M'_{32}, \quad L_2 = -M'_{31}, \quad L_3 = M'_{21},$$

$$P_a = M'_{4a} - M'_{a0}, \quad C_a = M'_{4a} + M'_{a0}, \quad (a = 1,2,3),$$

$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \quad X_k = P'_k \quad (k = 1,2,3), \quad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

2. The first-order differential equations in the space $\mathbf{M(1,3) \times R(u)}$

The group $P(1,4)$ acts on $M(1,3) \times R(u)$ (i.e. on the Cartesian product of the four-dimensional Minkowski space (of the independent variables x_0, x_1, x_2, x_3) and the number axis of the dependent variable u). The group $P(1,4)$ usually acts on $M(1,3) \times R(u)$ as a group generated by translations and rotations of this space.

Let

$$X = \sum_{i=0}^{3} \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u}$$

be one of the basis infinitesimal operators. The first prolongation of X has the form

$$X^{(1)} = X + \sum_{i=0}^{3}\left(\frac{\partial \eta}{\partial x_i} + \frac{\partial \eta}{\partial u}u_i - \sum_{j=0}^{3} u_j\frac{\partial \xi_j}{\partial x_i} - \sum_{j=0}^{3} u_i u_j\frac{\partial \xi_j}{\partial u}\right)\frac{\partial}{\partial u_i},$$

where $u_i \equiv \dfrac{\partial u}{\partial x_i}$, $i = 0, 1, 2, 3$.

Now, a function $J(x, u^{(1)})$ is a first-order differential invariant if

$$X^{(1)} \cdot J(x, u^{(1)}) = 0.$$

Here $u^{(1)} = (u, u_0, u_1, u_2, u_3)$ is an element of the first prolongation $R(u)^{(1)}$.

The group-theoretical methods (see, for example, [1, 2, 6]) enable us to construct new differential equations with non-trivial symmetry properties.

In our case these equations can be written in the following form:

$$F(J_1, J_2, \ldots, J_t) = 0,$$

where F is an arbitrary smooth function of its arguments, $\{J_1, J_2, \ldots, J_t\}$ is a functional basis of the first-order differential invariants of non-splitting subgroups of the group $P(1,4)$.

For all non-splitting subgroups of the group $P(1,4)$ we have described the first-order differential equations invariant under these subgroups and defined in the space $M(1,3) \times R(u)$.

Since we cannot present here all the results obtained, in this section we only give some of the new ones.

Below, for some non-splitting subgroups of the group $P(1,4)$, we write the basis elements of their respective Lie algebras and corresponding arguments $J_1, J_2, \ldots, J_t$ of the function F.

1. $\langle G + a_2 X_2,\ X_1,\ a_2 < 0 \rangle$,

$$J_1 = x_3, \quad J_2 = (x_0^2 - u^2)^{1/2},$$

$$J_3 = x_2 - a_2 \ln(x_0 + u),$$

$$J_4 = (x_0 + u)\frac{u_1}{u_0 + 1}, \quad J_5 = \frac{u_1}{u_2}, \quad J_6 = \frac{u_1}{u_3}, \quad J_7 = \frac{u_0^2 - 1}{u_1^2},$$

$$u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \ \mu = 0, 1, 2, 3;$$

2. $\langle G + aX_3,\ L_3 + dX_3,\ X_4,\ a < 0,\ d < 0 \rangle$,

$$J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = x_3 - a\ln(x_0 + u) + d \arctan\frac{x_1}{x_2},$$

$$J_3 = u_3 \frac{x_0 + u}{u_0 + 1},$$

$$J_4 = \frac{x_1 u_2 - x_2 u_1}{x_1 u_1 + x_2 u_2}, \quad J_5 = \frac{u_0^2 - 1}{u_3^2}, \quad J_6 = \frac{u_0^2 - 1}{u_1^2 + u_2^2};$$

3. $\langle G + a_1 X_1 + a_3 X_3,\ P_1,\ P_2,\ X_4,\ a_1 < 0,\ a_3 < 0 \rangle$,

$$J_1 = a_3 \ln(x_0 + u) - x_3,$$

$$J_2 = (x_0 + u)\frac{u_3}{u_0 + 1},$$

$$J_3 = x_2 + (x_0 + u)\frac{u_2}{u_0 + 1},$$

$$J_4 = a_3 \frac{x_0 + u}{u_0 + 1} u_1 + a_3 x_1 - a_1 x_3,$$

$$J_5 = \frac{u_0^2 - u_1^2 - u_2^2 - 1}{u_3^2};$$

4. $\langle L_3 + dX_3,\ P_3 + X_0,\ X_1,\ X_2,\ X_4,\ d < 0\rangle,$

$$J_1 = x_0 + u + \frac{u_3}{u_0 + 1},$$

$$J_2 = (x_0 + u)^2 - 2x_3 - 2d \arctan \frac{u_1}{u_2},$$

$$J_3 = \frac{u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2}, \quad J_4 = \frac{u_1^2 + u_2^2}{(u_0 + 1)^2};$$

5. $\langle L_3 + d_3 X_3,\ P_1,\ P_2,\ X_1,\ X_2,\ X_4,\ d_3 < 0\rangle,$

$$J_1 = x_0 + u, \quad J_2 = \frac{u_3}{u_0 + 1},$$

$$J_3 = \frac{u_1^2 + u_2^2}{(u_0 + 1)^2} + \frac{2}{u_0 + 1};$$

6. $\langle G + a_3 X_3,\ L_3 + d_3 X_3,\ P_1,\ P_2,\ X_1,\ X_2,\ X_4,\ a_3 < 0,\ d_3 < 0\rangle,$

$$J_1 = (x_0 + u)\frac{u_3}{u_0 + 1},$$

$$J_2 = \frac{u_0^2 - u_1^2 - u_2^2 - 1}{u_3^2}.$$

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