Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka IX, Częstochowa 2003

Abstract Boundedness and the Stability of the Pexider Equation

Roman Ger

Summary. Let $(S,+)$ be a semigroup (not necessarily Abelian) and let $(X,+)$ be a commutative group. We deal with an axiomatically given family $\mathcal{B} \subset 2^X$ of "bounded sets" and with mappings $f, g, h : S \longrightarrow X$ such that the transformation

$$S \times S \ni (x,y) \longmapsto f(x+y) - g(x) - h(y) \in X$$

remains $\mathcal{B}-$bounded.

Stability results existing in the literature in connection with the Pexider functional equation become special cases of our theorems up to the magnitude of approximating constants.

1. Introduction. The most prominent role in the theory of functional equations is played by the equation of Cauchy

$$\text{(C)} \qquad f(x+y) = f(x) + f(y)\,.$$

Its apparently far reaching generalization

$$\text{(P)} \qquad f(x+y) = g(x) + h(y)\,,$$

is commonly known as the Pexider functional equation and usually it can easily be reduced to (C) (see e.g. J. Aczél [1], J. Aczél & J. Dhombres [2] and M. Kuczma [7]). The celebrated Hyers-Ulam stability question: given an $\varepsilon \geq 0$ does any function f with values in a normed linear space $(X, \|\cdot\|)$ and satisfying the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \quad x, y \in S\,,$$

admit a homomorphism $a : S \longrightarrow X$ such that $f - a$ remains uniformly bounded in norm, has extensively been studied by many authors; see the monograph of D.H.Hyers, G.Isac and Th.M.Rassias [6] and the references therein.

Clearly, the corresponding stability question has also been asked in connection with the Pexider equation (P). Under various assumptions an affirmative answer was given by J.Tabor [9], K.Nikodem [8] and E.Głowacki & Z.Kominek [5] (also applied by J.Aczél [3]). All these authors reduce a problem in question to the stability of the Cauchy equation. Therefore, in what follows, dealing with the stability problem of the Pexider equation in a fairly general setting we shall restrict ourselves to such type reduction. Then we may consider the problem as solved as long as the exactness of approximation is disregarded. More precisely, assuming that the Pexider difference

$$(x,y) \ni S \times S \longmapsto f(x+y) - g(x) - h(y) \in X$$

is bounded (in a sense) we look for a homomorphism $A : S \longrightarrow X$ which would be uniformly close to each member of the triplet (f, g, h). The point is that neither commutativity of the domain nor the existence of a neutral element in S is assumed.

The results spoken of in this paper were presented by the author at the 31-st International Symposium on Functional Equations (Debrecen, Hungary, 1993); see [4].

2. Abstract boundedness. We begin with a definition of a general boundedness notion. In many instances, a set bounded in our sense may be pretty far from an intuitive feeling of boundedness. Nevertheless, classical boundedness notions in algebraic structures occurring in the literature prove to be special cases of the boundedness considered below.

DEFINITION. Let $(X, +)$ be a group and let $\mathcal{B} \subset 2^X$ be a set family such that

(i) $\mathcal{B}$ contains all singletons ;

(ii) $A + B \in \mathcal{B}$ for any two members A, B from $\mathcal{B}$;

(iii) $-B \in \mathcal{B}$ whenever $B \in \mathcal{B}$.

Then we say that the group $(X, +)$ is endowed with a $\mathcal{B}-$*boundedness structure* and the members of $\mathcal{B}$ along with all their subsets are termed $\mathcal{B}-$*bounded sets.*

Let $S \neq \emptyset$ be an arbitrary set and let $(X, +)$ be an Abelian group endowed with a $\mathcal{B}-$boundedness structure. A map $\varphi : S \longrightarrow X$ is called $\mathcal{B}-$*bounded* if and only if the set $\varphi(S)$ is $\mathcal{B}-$bounded.

Examples:

- $(X,+)$ - the additive group of a linear Hausdorff topological space X with $\mathcal{B}$ being the family of all bounded sets in X (in the regular sense).

- $(X,+)$ - the additive group of an infinite dimensional (real or complex) linear space X with

  $$\mathcal{B} := \{X_0 + c : X_0 \text{ is a finite dimensional linear subspace of } X,\ c \in X\}.$$

- $(X,+)$ - the additive group of an infinite dimensional (real or complex) linear space X; let X_0 be a linear subspace of X with infinite codimension. Put

  $$\mathcal{B} := \{C + c : X_0 \supset C \text{ is convex and } c \in X\}.$$

- $(X,+)$ - a non-compact Hausdorff topological group with the family $\mathcal{B}$ consisting of all compact subsets of X.

- $(X,+)$ - an uncountable group with the family $\mathcal{B}$ of all at most countable subsets of X.

- $(X,+)$ - the additive group of a vector lattice $(X,\preceq)$ with the totality $\mathcal{B}$ of all order bounded subsets of X.

- $(X,+)$ - a commutative group; let $(X,+)$ be a proper subgroup of $(X,+)$. Put

  $$\mathcal{B} := \{Z + c : Z \subset X_0,\ c \in X\}.$$

3. Stability. We begin with a definition that, roughly speaking, matches the pairs of semigroups and groups for which the Cauchy functional equation is stable in the sense of Hyers and Ulam.

Definition. Let $(S,+)$ be a semigroup and let $(X,+)$ be a group endowed with a $\mathcal{B}$–boundedness structure. We say that the pair (S,X) is $\mathcal{B}$–stable if and only if for every function $F : S \longrightarrow X$ such that the map

$$S \times S \ni (x,y) \longmapsto F(x+y) - F(x) - F(y) \in X$$

is $\mathcal{B}$–bounded there exists a homomorphism $A : S \longrightarrow X$ such that the function

$$S \ni x \longmapsto F(x) - A(x) \in X$$

is $\mathcal{B}$–bounded.

In what follows we shall permanently be using the fact that the sum of a finite number of $\mathcal{B}$-bounded functions is $\mathcal{B}$-bounded as well; in particular, the sum of a $\mathcal{B}$-bounded function and a constant one remains $\mathcal{B}$-bounded. Equally simple is the observation that whenever a function φ is $\mathcal{B}$-bounded so is the function $-\varphi$.

Theorem 1. Let $(S,+)$ be a semigroup and let $(X,+)$ be an Abelian group endowed with a $\mathcal{B}-$boundedness structure. Let $f,g,h : S \longrightarrow X$ be such that the map

$$S \times S \ni (x,y) \longmapsto f(x+y) - g(x) - h(y) \in X$$

is $\mathcal{B}-$bounded. If the pair (S,X) is $\mathcal{B}-$stable, then there exists a homomorphism $A : S \longrightarrow X$ such that for every $a \in S$ the functions

$$S \ni x \longmapsto f(a+x+a) - A(x) \in X$$
$$S \ni x \longmapsto g(a+x) - A(x) \in X$$
$$S \ni x \longmapsto h(x+a) - A(x) \in X$$

are all $\mathcal{B}-$bounded; such a homomorphism is unique modulo a $\mathcal{B}-$bounded function.

Proof. Fix arbitrarily an $a \in S$. Then the maps

$$S \ni x \longmapsto f(x+a) - g(x) \in X \tag{1}$$

and

$$S \ni y \longmapsto f(a+y) - h(y) \in X \tag{2}$$

are both $\mathcal{B}-$bounded whence so is also the map

$$S \times S \ni (x,y) \longmapsto f(x+y) - f(x+a) - f(a+y) \in X \tag{3}$$

because it coincides with

$$S \times S \ni (x,y) \longmapsto [f(x+y) - g(x) - h(y)]$$
$$+[g(x) - f(x+a)] + [h(y) - f(a+y)] \in X.$$

Setting now

$$F(x) := f(a+x+a), \quad x \in S,$$

and replacing x and y in (3) by $a+x$ and y+a, respectively, we infer that the Cauchy difference

$$S \times S \ni (x,y) \longmapsto F(x+y) - F(x) - F(y) \in X$$

is $\mathcal{B}$-bounded. Hence, since the pair (S, X) is $\mathcal{B}$-stable, there exists a solution
$A : S \longrightarrow X$ to the Cauchy functional equation (C) such that the difference

$$S \ni x \longmapsto F(x) - A(x) = f(a + x + a) - A(x) \in X$$

yields a $\mathcal{B}$-bounded function.

Now, since the map (1) is $\mathcal{B}$-bounded so is also the difference

$$S \ni x \longmapsto F(x) - g(a + x) = f(a + x + a) - g(a + x) \in X.$$

This implies that the difference

$$S \ni x \longmapsto g(a + x) - A(x) = [g(a + x) - F(x)] + [F(x) - A(x)] \in X.$$

remains $\mathcal{B}$-bounded.

Similarly, with the aid of the $\mathcal{B}$-boundedness of the map (2), we get the $\mathcal{B}$-boundedness of the difference

$$S \ni x \longmapsto h(x + a) - A(x) \in X.$$

To show that the homomorphism A does not depend upon the choice of a assume that for some $b \in S$ there exists a homomorphism $B : S \longrightarrow X$ such that the map

$$S \ni x \longmapsto f(b + x + b) - B(x) \in X$$

is $\mathcal{B}$-bounded. Since for every $x, y \in S$ one has

$$\begin{aligned} &f(x + a) - f(x + b) + f(a + y) - f(b + y) \\ &= [f(x + y) - f(x + b) - f(b + y)] - [f(x + y) - f(x + a) - f(a + y)] \end{aligned}$$

and the differences in square brackets remain $\mathcal{B}$-bounded because of (3), we infer that the map

$$S \ni x \longmapsto f(x + a) - f(x + b) \in X$$

is $\mathcal{B}$-bounded as well. Thus, for every $x \in S$, we obtain the $\mathcal{B}$-boundedness of the expression

$$\begin{aligned} &f(a + x + a) - f(b + x + b) = [f(a + x + a) - f(a) - f(x + a)] \\ &\quad -[f(b + x + b) - f(b) - f(x + b)] + [f(x + a) - f(x + b)] + [f(a) - f(b)] \end{aligned}$$

with respect to the variable $x \in S$. Consequently, the difference

$$A(x) - B(x) = [A(x) - f(a + x + a)]$$

$$+[f(a+x+a)-f(b+x+b)]+[f(b+x+b)-B(x)]$$

considered as a function of x is $\mathcal{B}$−bounded, too. Therefore, the functions

$$S \ni x \longmapsto f(b+x+b)-A(x) = [f(b+x+b)-B(x)]+[B(x)-A(x)] \in X,$$

$$S \ni x \longmapsto g(b+x)-A(x) = [g(b+x)-B(x)]+[B(x)-A(x)] \in X,$$

and

$$S \ni x \longmapsto h(x+b)-A(x) = [h(x+b)-B(x)]+[B(x)-A(x)] \in X,$$

are all $\mathcal{B}$−bounded which completes the proof.

Contrary to possible expectation, in general, a $\mathcal{B}$−bounded difference of two homomorphisms, which plainly is a homomorphism as well, need not be identically zero even in the case where a $\mathcal{B}$−boundedness structure in question is not the trivial one ($\mathcal{B} = 2^X$). Actually, take for example

$$(S,+) = (X,+) := (\mathbb{R},+) \text{ – the additive group of all reals}$$

and

$$\mathcal{B} := \{T \subset \mathbb{R} : T \text{ is at most countable}\}.$$

Then any additive surjection of $\mathbb{R}$ onto the field $\mathbb{Q}$ of all rationals (see e.g. M. Kuczma [7, p. 286]) yields a nonvanishing $\mathcal{B}$−bounded homomorphism between $\mathbb{R}$ and $\mathbb{Q} \subset \mathbb{R}$.

On the other hand, 0 is the only $\mathcal{B}$−bounded homomorphism provided that the corresponding $\mathcal{B}$−boundedness structure enjoyes the following property: for every set $B \in \mathcal{B}$ if $nu \in B$ for all positive integers n then $u = 0$.

Theorem 2. Under the assumptions of Theorem 1, if for some elements $a, b \in S$ the map

$$S \ni x \longmapsto f(a+x) - f(x+b) \in X$$

is $\mathcal{B}$−bounded, then there exists a homomorphism $A : S \longrightarrow X$ such that the maps

$$\begin{aligned} S+S \ \ni\ & x \longmapsto f(x) - A(x) \in X \\ S \ \ni\ & x \longmapsto g(x) - A(x) \in X \\ S \ \ni\ & x \longmapsto h(x) - A(x) \in X \end{aligned}$$

are all $\mathcal{B}$−bounded; such a homomorphism is unique modulo a $\mathcal{B}$−bounded function.

Conversely, for every homomorphism $A : S \longrightarrow X$ and any $\mathcal{B}$–bounded functions $p : S + S \longrightarrow X$ and $q, r : S \longrightarrow X$ the triplet

$$f := \begin{cases} A|_{S+S} + p \\ \text{arbitrary on } S \setminus (S+S) \end{cases}, \quad g := A + q, \quad h := A + r,$$

has a $\mathcal{B}$–bounded Pexider difference

$$S \times S \ni (x, y) \longmapsto f(x+y) - g(x) - h(y) \in X.$$

Proof. Note that the maps

$$S \ni x \longmapsto f(a+x) - g(a) - h(x) \in X$$

and

$$S \ni x \longmapsto f(x+b) - g(x) - h(b) \in X$$

are $\mathcal{B}$–bounded; therefore, so is their difference

$$S \ni x \longmapsto [g(x) - h(x)] + [f(a+x) - f(x+b)] + [h(b) - g(a)] \in X.$$

Consequently, the difference $g - h$ is $\mathcal{B}$–bounded which implies that the map

$$S \times S \ni (x, y) \longmapsto f(x+y) - g(x) - g(y) \in X$$

is $\mathcal{B}$–bounded, too. This proves that so are also the differences

$$S \times S \times S \ni (x, y, z) \longmapsto f(x+y+z) - g(x) - g(y+z) \in X$$

and

$$S \times S \times S \ni (x, y, z) \longmapsto f(x+y+z) - g(x+y) - g(z) \in X,$$

whence we deduce the $\mathcal{B}$–boundedness of the function

$$S \times S \times S \ni (x, y, z) \longmapsto g(x) + g(y+z) - g(x+y) - g(z) \in X.$$

In particular, setting here $x = y = a$, we get the $\mathcal{B}$–boundedness of the map

$$S \ni z \longrightarrow g(a+z) - g(z) \in X.$$

On account of Theorem 1, we get the existence of a homomorphism $A : S \longrightarrow X$ such that the difference

$$S \ni z \longrightarrow g(a+z) - A(z) \in X$$

remains $\mathcal{B}$–bounded. Consequently, so does also the map

$$S \ni z \longrightarrow g(z) - A(z) \in X$$

Now, since we know already that $h - g$ is $\mathcal{B}$–bounded, so is also the difference
$h - A = (h - g) + (g - A)$.

Finally, for all $x, y \in S$, one has

$$f(x+y) - A(x+y) = [f(x+y) - f(x+a) - f(a+y)] \\ +[f(x+a) - g(x)] + [g(x) - A(x)] + [f(a+y) - h(y)] + [h(y) - A(y)]$$

which, by means of (3), shows that the map

$$S + S \ni x \longmapsto f(x) - A(x) \in X$$

is $\mathcal{B}$–bounded.

Thus the proof has been completed because the uniqueness of A (up to a $\mathcal{B}$–bounded summand) as well as the latter assertion of the theorem are subjects for a straightforward verification.

4. Concluding remarks. In what follows we preserve the denotations and the assumptions of Theorem 1.

1. The additional condition occurring in the statement of Theorem 2 is automatically satisfied in each of the following situations:
   (a) $(S, +)$ is a monoid (take $a = b = 0$);
   (b) $(S, +)$ is commutative (take any $a = b$);
   (c) the center of $(S, +)$ is nonvoid (take any $a = b$ from the center);
   (d) there exist $a, b \in S$ such that $a + x = x + b$ for all $x \in S$.

2. If, for a single pair $(a, b) \in S \times S$, the function

$$S \ni x \longmapsto f(a+x) - f(x+b) \in X$$

   is unbounded, then there exists no additive map $A : S \longrightarrow X$ such that the difference $B := f - A$ were $\mathcal{B}$-bounded on $S + S$. Indeed, if B were $\mathcal{B}$-bounded for some homomorphism $A : S \longrightarrow X$, since for every x from S the points $a + x$ and $x + b$ belong to $S + S$, we would get

$$f(a+x) - f(x+b) = A(a+x) + B(a+x) \\ -A(x+b) - B(x+b) = B(a+x) - B(x+b) + A(a) - A(b),$$

- a contradiction, because the right hand side is $\mathcal{B}$-bounded whereas the left hand side is not.

3. To compare our results with those quoted in the Introduction, let us mention that J. Tabor [9] was assuming the domain $(S,+)$ to be a group with a special kind of weak commutativity, in K. Nikodem's paper [8] a commutative monoid was supposed to be the domain, whereas E. Głowacki & Z. Kominek [5] assumed that $(S,+)$ is a commutative semigroup. In all these papers, a kind of sequentially complete topological vector space stands for the target structure $(X,+)$.

References

[1] J. Aczél, *Lectures on Functional Equations and Their Applications*, Academic Press, New York, 1966.

[2] J. Aczél, J. Dhombres, *Some Aspects of Functional Equations*, Cambridge Univ. Press, Cambridge, 1989.

[3] J. Aczél, *A simple functional equations method applied to several problems in the social sciences*, Talk presented at the Silesian University of Katowice (Poland), April 10, 1992.

[4] R. Ger, *Abstract boundedness and the stability of the Pexider equation*, Talk presented at the 31-st International Symposium on Functional Equations, August 22 - 28, 1993, Debrecen (Hungary); Report of Meeting: Aequationes Math. **47**, 275, 1994.

[5] E. Głowacki, Z. Kominek, *On stability of the Pexider equation on semigroups,* A Volume dedicated to D. H. Hyers & S. Ulam, Hadronic Press, Palm Harbor, FL, 111-116, 1994.

[6] D. H. Hyers, G. Isac, Th. M. Rassias, *Stability of Functional Equations in Several Variables,* Birkhäuser, Boston-Basel-Berlin, 1998.

[7] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, PWN & Uniwersytet Śląski, Warszawa-Kraków-Katowice, 1985.

[8] K. Nikodem, *The stability of the Pexider equation,* Ann. Math. Sil. **5**, 91-93, 1991.

[9] J. Tabor, *Restricted stability of the Cauchy and the Pexider equations,* Zeszyty Nauk. Politech. Sl., Ser. Mat.-Fiz. **64**, 203-221, 1990

Institute of Mathematics and Computer Science,
Pedagogical University of Częstochowa,
al. Armii Krajowej 13/15,
42-201 Częstochowa,
Poland
e-mail: r.ger@wsp.czest.pl