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The Proof of Certain Tarski's Theorem about the Existence of One-element Base for Axiomatizable Systems of Propositional Calculus

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In [1] the following theorem relating the existence of one-element base for spacious class of axiomatizable propositional calculus has been given:

Theorem 1. System L, as well as each axiomatizable system of propositional calculus, contains sentences „$CpCqp$" and „$CpCqCCpCqrr$" (or „$CpCqCCpCqrCsr$"), possesses the base consisting of only one sentence [1].

In Postscript added to the English translation of publication [1][2] the outline of proof of the above theorem, found by R. McKenzie, has been given.

Author of the article advises to give the full proof of Theorem 1, because the outline contained in Postscript does not contain essential reasonings for the proof.

Let S be a set of all variables of sentences $v_0, v_1, v_2, \ldots, v_n, \ldots$ and all expressions formed out of these variables by implication connective[3].

Let us suppose that any axiomatizable system M has a complete system of axioms $A = \{a_0, a_1, a_2, \ldots, a_n\}$ (for certain $n \in N$), where

$$a_0 = p \Rightarrow (q \Rightarrow p), \; a_1 = p \Rightarrow (q \Rightarrow [(p \Rightarrow (q \Rightarrow r)) \Rightarrow r]).$$

Let

$$(1) \qquad f(x,y) = [[v_k \Rightarrow (v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x))] \Rightarrow (y \Rightarrow v_{k+3})] \Rightarrow v_{k+3},$$

where v_k is a variable of sentence with the smallest index k, which appears neither in x nor in y.

[1]Symbol L marks usual two-valued system of propositional calculus.

[2]The translation is in [2] on pp. 59. Polish version of [1] and Postscript can be found in monograph [3] on pp. 3-30.

[3]We will use usual symbolics, with use of parentheses, instead of Polish without parentheses symbolics.

We are going to prove that [4]:

$$x \in Cn(\{f(x,y)\}), \tag{2}$$

$$y \in Cn(\{f(x,y), a_0\}), \tag{3}$$

$$f(x,y) \in Cn(\{x, y, a_0, a_1\}). \tag{4}$$

The proof of formula (2):

1. $f(x,y) \in Cn(\{f(x,y)\})$ {it's obvious}
2. $[[v_k \Rightarrow (v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x))] \Rightarrow (y \Rightarrow v_{k+3})] \Rightarrow v_{k+3} \in \in Cn(\{f(x,y)\})$ $\{1, (1)\}$
3. $W = \{[[v_k \Rightarrow (y \Rightarrow (y \Rightarrow x))] \Rightarrow (y \Rightarrow (y \Rightarrow x))] \Rightarrow (y \Rightarrow x)\} \Rightarrow x \in \in Cn(\{f(x,y)\})$
   $\{2 : v_k | v_k \Rightarrow (y \Rightarrow (y \Rightarrow x)),\ v_{k+1}|y,\ v_{k+2}|y,\ v_{k+3}|x\}$
4. $U = [[v_k \Rightarrow (y \Rightarrow (y \Rightarrow x))] \Rightarrow (y \Rightarrow (y \Rightarrow x))] \Rightarrow (y \Rightarrow x) \in \in Cn(\{f(x,y)\})$ $\{2 : v_{k+1}|y,\ v_{k+2}|y,\ v_{k+3}|y \Rightarrow x\}$

   $x \in Cn(\{f(x,y)\})$ $\{MP : 3,\ 4\}$

The proof of formula (3):

1. $f(x,y) \in Cn(\{f(x,y)\})$ {it's obvious}
2. $[[v_k \Rightarrow (v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x))] \Rightarrow (y \Rightarrow v_{k+3})] \Rightarrow v_{k+3} \in \in Cn(\{f(x,y)\})$ $\{1, (1)\}$
3. $a_0 = (p \Rightarrow (q \Rightarrow p)) \in Cn(\{a_0\})$ {it's obvious}
4. $y \Rightarrow (y \Rightarrow y) \in Cn(\{a_0\})$ $\{3 :\ p|y,\ q|y\}$
5. $(y \Rightarrow (y \Rightarrow y)) \Rightarrow [[v_k \Rightarrow (v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x))] \Rightarrow (y \Rightarrow (y \Rightarrow y))] \in \in Cn(\{a_0\})$ $\{3 : p|y \Rightarrow (y \Rightarrow y),\ q|v_k \Rightarrow (v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x))\}$

[4] Consequence function Cn is based on the substitution rule and the detachment one.

6. $[v_k \Rightarrow (v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x))] \Rightarrow (y \Rightarrow (y \Rightarrow y)) \in Cn(\{a_0\}) \subseteq$
$\subseteq Cn(\{f(x,y),\ a_0\})$ $\{MP: 5,\ 4\}$

7. $\{[v_k \Rightarrow (v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x))] \Rightarrow (y \Rightarrow (y \Rightarrow y))\} \Rightarrow (y \Rightarrow y) \in$
$\in Cn(\{f(x,y)\}) \subseteq Cn(\{f(x,y),\ a_0\})$ $\{2:\ v_{k+3}|y \Rightarrow y\}$

8. $y \Rightarrow y \in Cn(\{f(x,y), a_0\})$ $\{MP: 6,\ 7\}$

9. $(y \Rightarrow y) \Rightarrow \{[v_k \Rightarrow (v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x))] \Rightarrow (y \Rightarrow y)\} \in Cn(\{a_0\}) \subseteq$
$\subseteq Cn(\{f(x,y), a_0\})$ $\{3: p|y \Rightarrow y,\ q|v_k \Rightarrow (v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x))\}$

10. $[v_k \Rightarrow (v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x))] \Rightarrow (y \Rightarrow y) \in Cn(\{f(x,y), a_0\})$
$\{MP: 9,\ 8\}$

11. $\{[v_k \Rightarrow (v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x))] \Rightarrow (y \Rightarrow y)\} \Rightarrow y \in Cn(\{f(x,y)\}) \subseteq$
$\subseteq Cn(\{f(x,y), a_0\})$ $\{2: v_{k+3}|y\}$

$y \in Cn(\{f(x,y), a_0\})$ $\{MP: 11,\ 10\}$

The proof of formula (4):

1. $a_0 = p \Rightarrow (q \Rightarrow p) \in Cn(\{a_0\})$ {it's obvious}

2. $x,\ x \Rightarrow (v_{k+2} \Rightarrow x) \in Cn(\{x, a_0\})$ $\{1: p|x,\ q|v_{k+2}\}$

3. $v_{k+2} \Rightarrow x \in Cn(\{x, a_0\})$ $\{MP: 2\}$

4. $(v_{k+2} \Rightarrow x) \Rightarrow [v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x)] \in Cn(\{a_0\}) \subseteq Cn(\{x, a_0\})$
$\{1: p|v_{k+2} \Rightarrow x,\ q|v_{k+1}\}$

5. $v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x) \in Cn(\{x, a_0\})$ $\{MP: 4, 3\}$

6. $[v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x)] \Rightarrow \{v_k \Rightarrow [v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x)]\} \in Cn(\{a_0\}) \subseteq$
$\subseteq Cn(\{x, a_0\})$ $\{1: p|v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x),\ q|v_k\}$

7. $v_k \Rightarrow [v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x)] \in Cn(\{x, a_0\})$ $\{MP: 6, 5\}$

8. $a_1 = p \Rightarrow (q \Rightarrow [(p \Rightarrow (q \Rightarrow r)) \Rightarrow r]) \in Cn(\{a_1\})$ {it's obvious}

9. $[v_k \Rightarrow [v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x)]] \Rightarrow \{y \Rightarrow [([v_k \Rightarrow [v_{k+1} \Rightarrow (v_{k+2} \Rightarrow$
$\Rightarrow x)]] \Rightarrow (y \Rightarrow v_{k+3})) \Rightarrow v_{k+3}]\} \in Cn(\{a_1\}) \subseteq Cn(\{x, a_0, a_1\})$
$\{8: p|v_k \Rightarrow [v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x)],\ q|y,\ r|v_{k+3}\}$

10. $y \Rightarrow [([v_k \Rightarrow [v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x)]] \Rightarrow (y \Rightarrow v_{k+3})) \Rightarrow v_{k+3}] \in$
$\in Cn(\{x, a_0, a_1\})$ $\{MP: 9, 7\}$

11. $y \in Cn(\{y\}) \subseteq Cn(\{x, y, a_0, a_1\})$ {it's obvious}

12. $[[v_k \Rightarrow (v_{k+1} \Rightarrow (v_{k+2} \Rightarrow x))] \Rightarrow (y \Rightarrow v_{k+3})] \Rightarrow v_{k+3} \in$ $\in Cn(\{x, y, a_0, a_1\})$ $\{MP : 10, 11\}$

$f(x, y) \in Cn(\{x, y, a_0, a_1\})$ {12, (1)}

Let us assume the following recurrence definition of complete sequence of expressions $b_0,\ b_1,\ b_2, \ldots,\ b_n$:

(5) $b_0 = a_0$ and $b_{k+1} = f(b_k, a_{k+1})$ for every $k < n,\ k \in N$.

With the help of complete induction with respect to k , we are going to prove, that:

(6) $$b_k \in Cn(\{a_0, a_1, \ldots, a_k\}),$$

(7) $$a_0, a_1, \ldots, a_k \in Cn(\{b_k\})\ for\ k = 0, 1, \ldots, n.$$

The proof of formula (6):
One should prove, that $(\forall\ 0 \leq k \leq n)\quad b_k \in Cn(\{a_0, a_1, \ldots, a_k\})$.
If $k = 0$ then on the basis of formula (5) and property $a_0 \in Cn(\{a_0\})$ we obtain $b_0 \in Cn(\{a_0\})$.
Let us make an inductive assumption:

$$(\forall\ 0 \leq k < n)\ \ b_k \in Cn(\{a_0, a_1, \ldots, a_k\})$$

that is the assumption

(a) $$b_i \in Cn(\{a_0, a_1, \ldots, a_i\}),\ \text{for}\ i = 0, 1, \ldots, n-1.$$

Let $i = n$.
For $i = n - 1$ on the ground of formula (5) we obtain $b_n = f(b_{n-1}, a_n)$, however
$f(b_{n-1}, a_n) \in Cn(\{b_{n-1}, a_n, a_0, a_1\})$ (see formula (4)), therefore

(b) $$b_n \in Cn(\{b_{n-1}, a_n, a_0, a_1\}).$$

From the inductive assumption (formula (a)) it results that

(c) $$b_{n-1} \in Cn(\{a_0, a_1, \ldots, a_{n-1}\}).$$

From formula (c) it results that

$$Cn(\{b_{n-1}, a_n, a_0, a_1\}) \subseteq Cn(\{a_0, a_1, \ldots, a_{n-1}, a_n\}),$$

on the basis of formula (b) we get :

$$b_n \in Cn(\{a_0, a_1, \ldots, a_{n-1}, a_n\}).$$

The proof of formula (7):

For $k = 0$ on the basis of formula (5) and $b_0 \in Cn(\{b_0\})$ we obtain $a_0 \in Cn(\{b_0\})$.

Let us make an inductive assumption:

$$a_0, a_1, \ldots, a_i \in Cn(\{b_i\}), \text{ for } i = 1, 2, \ldots, n-1.$$

Let $i = n$.

On the ground of formula (5) we obtain

$$Cn(\{b_n\}) = Cn(\{f(b_{n-1}, a_n)\}).$$

Because $b_{n-1} \in Cn(\{b_{n-1}\}) \subseteq Cn(\{f(b_{n-1}, a_n), b_{n-1}\})$, so on the basis of inductive assumption we have

(d) $$a_0, a_1, \ldots, a_{n-1} \in Cn(\{f(b_{n-1}, a_n), b_{n-1}\}).$$

From formula (3) it results that

(e) $$a_n \in Cn(\{f(b_{n-1}, a_n), a_0\}) \subseteq Cn(\{f(b_{n-1}, a_n), a_0, b_{n-1}\}).$$

Because, on the basis of (d) :

$$a_0 \in Cn(\{f(b_{n-1}, a_n), b_{n-1}\}),$$

so

$$Cn(\{f(b_{n-1}, a_n), a_0, b_{n-1}\}) \subseteq Cn(\{f(b_{n-1}, a_n), b_{n-1}\}),$$

on the basis of (e) we obtain:

(f) $$a_n \in Cn(\{f(b_{n-1}, a_n), b_{n-1}\}).$$

From formula (5) we have $b_n = f(b_{n-1}, a_n)$, so from formulas (d) and (f) it results that :

$$a_0, a_1, \ldots, a_{n-1}, a_n \in Cn(\{b_{n-1}, b_n\}).$$

From formula (2) we have $b_{n-1} \in Cn(\{f(b_{n-1}, a_n)\})$, that is

$$b_{n-1} \in Cn(\{b_n\}).$$

Finally,

$$a_0, a_1, \ldots, a_{n-1}, a_n \in Cn(\{b_n\}),$$

what finishes the proof of formula (7).

Assuming in formula (7) $k = n$ we can say that the set $\{b_n\}$ is a base of system M.

We are going to prove the second part of theorem 1.

Let us suppose that any axiomatizable system M has a complete system of axioms $A = \{a_0, a_1^*, a_2, \ldots, a_n\}$ (for certain $n \in N$), where

$$a_0 = p \Rightarrow (q \Rightarrow p), \; a_1^* = p \Rightarrow \{q \Rightarrow [(p \Rightarrow (q \Rightarrow r)) \Rightarrow (s \Rightarrow r)]\}.$$

Let

$$(8) \qquad g(x, y) = \{[v_k \Rightarrow (v_{k+1} \Rightarrow x)] \Rightarrow (y \Rightarrow v_{k+3})\} \Rightarrow (v_{k+2} \Rightarrow v_{k+3}),$$

where v_k is a variable of sentence with the smallest index k, which appears neither in x nor in y.
We are going to prove that :

$$(9) \qquad x \in Cn(\{g(x, y)\}),$$

$$(10) \qquad y \in Cn(\{g(x, y), a_0\}),$$

$$(11) \qquad g(x, y) \in Cn(\{x, y, a_0, a_1^*\}).$$

The proof of formula (9):

1. $g(x, y) \in Cn(\{g(x, y)\})$ {it's obvious}

2. $\{[v_k \Rightarrow (v_{k+1} \Rightarrow x)] \Rightarrow (y \Rightarrow v_{k+3})\} \Rightarrow (v_{k+2} \Rightarrow v_{k+3}) \in$
$\in Cn(\{g(x,y)\})$ $\{1, (8)\}$

3. $\{[v_k \Rightarrow (y \Rightarrow x)] \Rightarrow (y \Rightarrow x)\} \Rightarrow (v_{k+2} \Rightarrow x) \in$
$\in Cn(\{g(x,y)\})$ $\{2 : v_{k+3}|x,\ v_{k+1}|y\}$

4. $W^* = \{[[v_k \Rightarrow (v_{k+1} \Rightarrow x)] \Rightarrow (y \Rightarrow x)] \Rightarrow (y \Rightarrow x)\} \Rightarrow$
$\Rightarrow (g(x,y) \Rightarrow x) \in Cn(\{g(x,y)\})$

$\{3 : v_k | v_k \Rightarrow (v_{k+1} \Rightarrow x),\ v_{k+2} | f(x,y)\}$

5. $U^* = \{[v_k \Rightarrow (v_{k+1} \Rightarrow x)] \Rightarrow (y \Rightarrow x)\} \Rightarrow (y \Rightarrow x) \in$
$\in Cn(\{g(x,y)\})$ $\{2 : v_{k+2}|y,\ v_{k+3}|x\}$

6. $(g(x,y) \Rightarrow x) \in Cn(\{g(x,y)\})$ $\{MP : 4,\ 5\}$

$x \in Cn(\{g(x,y)\})$ $\{MP : 1,\ 6\}$

The proof of formula (10):

1. $g(x,y) \in Cn(\{g(x,y)\})$ {it's obvious}

2. $\{[v_k \Rightarrow (v_{k+1} \Rightarrow x)] \Rightarrow (y \Rightarrow v_{k+3})\} \Rightarrow (v_{k+2} \Rightarrow v_{k+3}) \in$
$\in Cn(\{g(x,y)\})$ $\{1, (8)\}$

3. $a_0 = (p \Rightarrow (q \Rightarrow p)) \in Cn(\{a_0\})$ {it's obvious}

4. $y \Rightarrow (y \Rightarrow y) \in Cn(\{a_0\})$ $\{3 :\ p|y,\ q|y\}$

5. $[y \Rightarrow (y \Rightarrow y)] \Rightarrow \{[v_k \Rightarrow (v_{k+1} \Rightarrow x)] \Rightarrow [y \Rightarrow (y \Rightarrow y)]\} \in$
$\in Cn(\{a_0\})$ $\{3 : p|y \Rightarrow (y \Rightarrow y),\ q|v_k \Rightarrow (v_{k+1} \Rightarrow x)\}$

6. $[v_k \Rightarrow (v_{k+1} \Rightarrow x)] \Rightarrow [y \Rightarrow (y \Rightarrow y)] \in Cn(\{a_0\}) \subseteq$
$\subseteq Cn(\{g(x,y),\ a_0\})$ $\{MP : 5,\ 4\}$

7. $\{[v_k \Rightarrow (v_{k+1} \Rightarrow x)] \Rightarrow [y \Rightarrow (y \Rightarrow y)]\} \Rightarrow [v_{k+2} \Rightarrow (y \Rightarrow y)] \in$
$\in Cn(\{g(x,y)\}) \subseteq Cn(\{g(x,y),\ a_0\})$ $\{2 :\ v_{k+3}|y \Rightarrow y\}$

8. $v_{k+2} \Rightarrow (y \Rightarrow y) \in Cn(\{g(x,y), a_0\})$ $\{MP : 6,\ 7\}$

9. $a_0 \Rightarrow (y \Rightarrow y) \in Cn(\{g(x,y), a_0\})$ $\{8 : v_{k+2}|a_0\}$

10. $y \Rightarrow y \in Cn(\{g(x,y), a_0\})$ $\{MP : 3,\ 9\}$

11. $(y \Rightarrow y) \Rightarrow \{[v_k \Rightarrow (v_{k+1} \Rightarrow x)] \Rightarrow (y \Rightarrow y)\} \in Cn(\{a_0\}) \subseteq$
$\subseteq Cn(\{g(x,y), a_0\})$ $\{3 : p|y \Rightarrow y,\ q|v_k \Rightarrow (v_{k+1} \Rightarrow x)\}$

12. $[v_k \Rightarrow (v_{k+1} \Rightarrow x)] \Rightarrow (y \Rightarrow y) \in Cn(\{g(x,y), a_0\})$
$\{MP: 10,\ 11\}$

13. $\{[v_k \Rightarrow (v_{k+1} \Rightarrow x)] \Rightarrow (y \Rightarrow y)\} \Rightarrow (a_0 \Rightarrow y) \in Cn(\{g(x,y)\}) \subseteq$
$\subseteq Cn(\{g(x,y), a_0\})$ $\{2: v_{k+2}|a_0,\ v_{k+3}|y\}$

14. $a_0 \Rightarrow y \in Cn(\{g(x,y), a_0\})$ $\{MP: 12,\ 13\}$

$y \in Cn(\{g(x,y), a_0\})$ $\{MP: 3,\ 14\}$

The proof of formula (11):

1. $a_0 = p \Rightarrow (q \Rightarrow p) \in Cn(\{a_0\}) \subseteq Cn(\{x, y, a_0, a_1^*\})$ {it's obvious}

2. $x \in Cn(\{x\}) \subseteq Cn(\{x, y, a_0, a_1^*\})$ {it's obvious}

3. $x \Rightarrow (v_{k+1} \Rightarrow x) \in Cn(\{a_0\}) \subseteq Cn(\{x, y, a_0, a_1^*\})$
$\{1: p|x,\ q|v_{k+1}\}$

4. $v_{k+1} \Rightarrow x \in Cn(\{x, y, a_0, a_1^*\})$ $\{MP: 2, 3\}$

5. $(v_{k+1} \Rightarrow x) \Rightarrow [v_k \Rightarrow (v_{k+1} \Rightarrow x)] \in Cn(\{x, y, a_0, a_1^*\})$
$\{1: p|v_{k+1} \Rightarrow x,\ q|v_k\}$

6. $v_k \Rightarrow (v_{k+1} \Rightarrow x) \in Cn(\{x, y, a_0, a_1^*\})$ $\{MP: 4, 5\}$

7. $a_1^* = p \Rightarrow \{q \Rightarrow [(p \Rightarrow (q \Rightarrow r)) \Rightarrow (s \Rightarrow r)]\} \in Cn(\{a_1^*\}) \subseteq$
$\subseteq Cn(\{x, y, a_0, a_1^*\})$ {it's obvious}

8. $[v_k \Rightarrow [v_{k+1} \Rightarrow x)] \Rightarrow \{y \Rightarrow [([v_k \Rightarrow v_{k+1} \Rightarrow x)] \Rightarrow (y \Rightarrow$
$\Rightarrow v_{k+3})) \Rightarrow (v_{k+2} \Rightarrow v_{k+3})]\} \in Cn(\{x, y, a_0, a_1^*\})$
$\{7: p|v_k \Rightarrow (v_{k+1} \Rightarrow x),\ q|y,\ s|v_{k+2},\ r|v_{k+3}\}$

9. $y \Rightarrow [([v_k \Rightarrow (v_{k+1} \Rightarrow x)] \Rightarrow (y \Rightarrow v_{k+3})) \Rightarrow (v_{k+2} \Rightarrow v_{k+3})] \in$
$\in Cn(\{x, y, a_0, a_1^*\})$ $\{MP: 6,8\}$

10. $y \in Cn(\{y\}) \subseteq Cn(\{x, y, a_0, a_1^*\})$ {it's obvious}

11. $\{[v_k \Rightarrow (v_{k+1} \Rightarrow x)] \Rightarrow (y \Rightarrow v_{k+3})\} \Rightarrow (v_{k+2} \Rightarrow v_{k+3}) \in$
$\in Cn(\{x, y, a_0, a_1^*\})$ $\{MP: 9, 10\}$

$$g(x,y) \in Cn(\{x,y,a_0,a_1^*\}) \qquad \{11,\ (8)\}$$

Let us assume the following recurrence definition of complete sequence of expressions $b_0,\ b_1,\ b_2, \ldots,\ b_n$:

(12) $$b_0 = a_0,\ b_1 = g(b_0, a_1^*) \text{ and } b_{k+1} = g(b_k, a_{k+1})$$
$$\text{for every } 1 \leq k < n,\ k \in N.$$

With the help of complete induction with respect to k , it can be proved that:

(13) $$b_k \in Cn(\{a_0, a_1^*, \ldots, a_k\}),$$

(14) $$a_0, a_1^*, \ldots, a_k \in Cn(\{b_k\})\ for\ k = 0, 1, \ldots, n.$$

The proofs of formulas (13) and (14) are similar to proofs of formulas (6) and (7), that's why they are passed over. The main difficulty in the proof of the second part of Theorem 1 concerning axioms a_0 and a_1^* consists in finding suitable function g(x,y).

References

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[2] A.Tarski, *Logic, Semantics, Metamathematics,* Papers from 1923 to 1938, Clarendon Press, Oxford, 1956, XIV+471 p.

[3] A.Tarski, *Pisma logiczno-filozoficzne*, tom 2, Metalogika, (Tłumaczenie i redakcja Jan Zygmunt), PWN, Warszawa, 2001, XIV+516 s.

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