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Nonlocality Versus Friction

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It is well known that small oscillations are descibed by the following ordinary differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = p \cos \omega_1 t \tag{1}$$

with y being the deviation from the equilibrium position, t the time, ω the frequency of natural oscillations, ω_1 and p the frequency and the reduced amplitude of the external force.

For example, for a spring and an electrical circuit we have, respectively

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{and} \qquad \omega = \frac{1}{\sqrt{LC}},$$

where m is mass, k is the stiffness coefficient, L and C are the electricity induction and the electricity capacity.

The particular solution of equation (1) has the form of oscillation

$$y = A \cos \omega_1 t \tag{2}$$

with the amplitude

$$A = \frac{p}{|\omega^2 - \omega_1^2|}. \tag{3}$$

The dependence of the nondimensional amplitude $A\omega^2/p$ on the nondimensional external force frequency ω_1/ω is shown in Fig. 1.

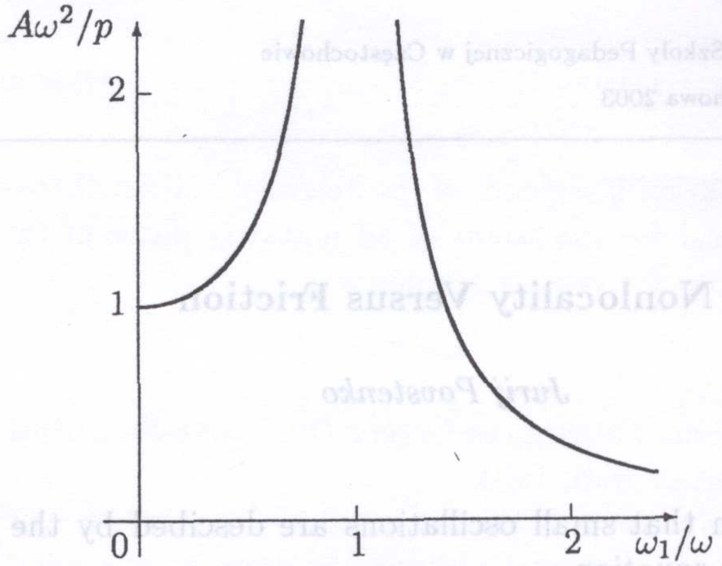

Fig. 1.

If $\omega_1 = \omega$, we obtain the infinitely large deviation (the resonance). Of course, this physically incorrect result is a consequence of the simplest mathematical model. In reality there is friction (mechanical friction for a spring or electricity resistance for an electrical circuit). Hence, we must consider the more precise mathematical model and obtain the more precise equation instead of (1). For instance, denoting the reduced friction coefficient by δ we obtain

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \delta \frac{\mathrm{d}y}{\mathrm{d}t} + \omega^2 y = p \cos \omega_1 t \tag{4}$$

having the particular solution

$$y = A \cos(\omega_1 t - \varphi) \tag{5}$$

with

$$A = \frac{p}{\sqrt{(\omega^2 - \omega_1^2)^2 + \delta^2 \omega_1^2}}, \qquad \operatorname{tg} \varphi = \frac{\delta \omega_1}{\omega^2 - \omega_1^2}. \tag{6}$$

The plot of $A\omega^2/p$ calculated from (6) is displayed in Fig. 2 for $\delta/\omega = 1$.

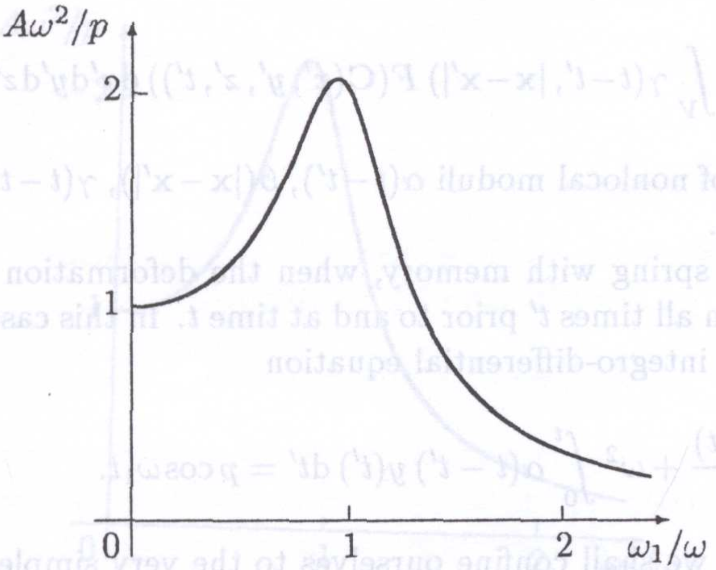

Fig. 2.

In many cases considering the complicated physical problem we obtain solutions having nonphysical singularities, but it is sometimes difficult to introduce "friction" to avoid them. In these cases we can consider the nonlocal theory. The goal of this paper is to show that nonlocality acts quite similar to friction.

Remind that the local dependence of one physical quantity (an effect $\mathbf{E}$) at a point $\mathbf{x}$ at time t ($\mathbf{E}(x,y,z,t)$) on a field of another physical quantity (a cause $\mathbf{C}$) at the same point $\mathbf{x}$ and at the same time t ($\mathbf{C}(x,y,z,t)$) has the following form

$$\mathbf{E}(x,y,z,t) = F(\mathbf{C}(x,y,z,t)). \tag{7}$$

For materials with memory (with time non-locality) the effect $\mathbf{E}$ at a point $\mathbf{x}$ at time t depends on the histories of causes at a point $\mathbf{x}$ at all past and present times

$$\mathbf{E}(x,y,z,t) = \int_{-\infty}^{t} \alpha(t-t') F(\mathbf{C}(x,y,z,t'))\,\mathrm{d}t'. \tag{8}$$

Space nonlocality means that the effect $\mathbf{E}$ at a point $\mathbf{x}$ at time t depends on causes at all points $\mathbf{x}'$ at time t

$$\mathbf{E}(x,y,z,t) = \int_V \beta(|\mathbf{x}-\mathbf{x}'|) F(\mathbf{C}(x',y',z',t))\,\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'. \tag{9}$$

When memory effect is accompanied by space non-locality we have dependence of the effect $\mathbf{E}$ at a point $\mathbf{x}$ at time t on causes at all points $\mathbf{x}'$

and at all times t' prior to and at time t

$$\mathbf{E}(x,y,z,t) = \int_{-\infty}^{t} \int_V \gamma(t-t', |\mathbf{x}-\mathbf{x}'|)\, F(\mathbf{C}(x',y',z',t'))\, \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\,\mathrm{d}t'. \quad (10)$$

The properties of nonlocal moduli $\alpha(t-t')$, $\beta(|\mathbf{x}-\mathbf{x}'|)$, $\gamma(t-t', |\mathbf{x}-\mathbf{x}'|)$ are descibed in [1-3].

Now consider a spring with memory, when the deformation force depends on deviation in all times t' prior to and at time t. In this case, instead of (1) we obtain the integro-differential equation

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \omega^2 \int_0^t \alpha(t-t')\, y(t')\,\mathrm{d}t' = p\cos\omega_1 t. \quad (11)$$

In the following we shall confine ourselves to the very simple nonlocal modulus

$$\alpha(t-t') = \begin{cases} \dfrac{\pi}{2T}\cos\dfrac{\pi(t-t')}{2T} & 0 \le t-t' \le T \\ 0 & t-t' \ge T \end{cases}$$

where T is the internal characteristic time. Eqution (11) is rewritten as

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \frac{\pi\omega^2}{2T}\int_0^T \cos\frac{\pi t'}{2T}\, y(t-t')\,\mathrm{d}t' = p\cos\omega_1 t. \quad (12)$$

It can be shown that the particular solution of (12) has the form

$$y = A\cos(\omega_1 t - \varphi) \quad (13)$$

with

$$A = \frac{p\Delta}{\omega_2\sqrt{\left[\cos\omega_1 T - \Delta\left(\dfrac{\omega_1}{\omega}\right)^2\right]^2 + \left[\dfrac{2T\omega_1}{\pi} - \sin\omega_1 T\right]^2}}, \quad (14)$$

$$\operatorname{tg}\varphi = \frac{\dfrac{2T\omega_1}{\pi} - \sin\omega_1 T}{\cos\omega_1 T - \Delta\left(\dfrac{\omega_1}{\omega}\right)^2}, \qquad \Delta = 1 - \left(\frac{2T\omega_1}{\pi}\right)^2.$$

The dependence of $A\omega^2/p$ on ω_1/ω is depicted in Fig. 3 for $2T\omega/\pi = 0.5$.

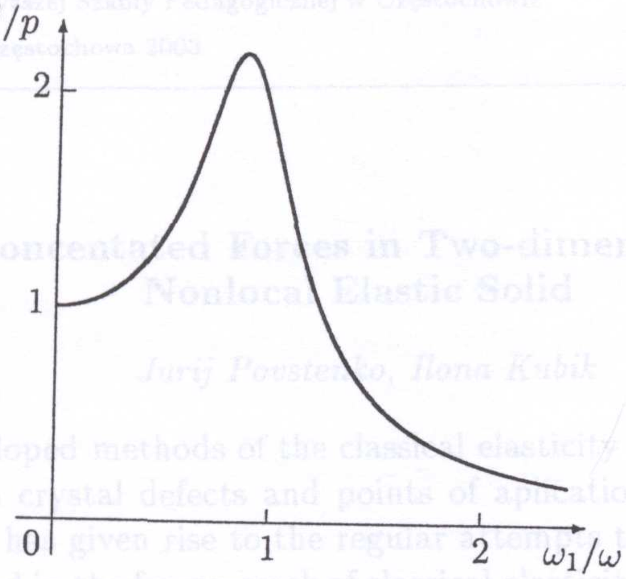

Fig. 3.

The comparison of Fig. 2 and Fig. 3 proves that in the case under consideration nonlocality acts quite similar to friction indeed. Other applications of nonlocal theory can be found in [4, 5].

References

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