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Concentated Forces in Two-dimensional Nonlocal Elastic Solid

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Well–developed methods of the classical elasticity break down at small distances from crystal defects and points of aplication of singular forces. This situation has given rise to the regular attempts to improve elastic solutions obtained in the frame–work of classical elasticity, for instant, combining elastic and discrete approaches for better description of high distorted regions.

In the past two decades, a considerable research efforts has been expended to develop the nonlocal theory of elasticity and to solve various problems of continuum mechanics using this theory. The essentials of the nonlocal theory were established by Kröner [1], Pidstryhach [2], Eringen [3], Edelen [4], Kunin [5] and others. Starting from interrelated equations describing elasticity and diffusion Pidstryhach [2] excluded the chemical potential from the constitutive equation for the stress tensor and obtained the nonlocal stress–strain relation. Kröner [1] and Kunin [5] started from discrete lattice and interpolated functions of discrete argument by special continuous functions. The stress–strain relation obtained in such a quasi-continuum is non–local.

The nonlocal theory reduces to the classical theory of elasticity in the long–wavelength limit and to the atomic lattice theory in the short–wavelength limit. The nonlocal theory is effective in removing non–physical singularities occurring at dislocations, disclinations, points of applications of singular forces, cracks, etc.

A number of problems solved in the frame–work of nonlocal theory indicate the power of the theory. It manifests some new physical phenomena and overcomes difficulties in classical theory such as classical singularities. Using this theory one can obtain more justified results.

In local elasticity the stress tensor $\boldsymbol{\sigma}$ satisfies the equilibrium equation

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = 0, \tag{1}$$

and is connected with the strain tensor $\mathbf{e}$ by the Hooke law

$$\boldsymbol{\sigma} = \lambda \operatorname{tr} \mathbf{e}\, \mathbf{I} + 2\mu \mathbf{e}, \tag{2}$$

where ∇ is the gradient operator, $\mathbf{f}$ is the body force vector, λ and μ are Lamé constants, $\mathbf{I}$ denotes the unit tensor.

Consider a concentrated force

$$\mathbf{f} = Q_y\,\delta(x)\,\delta(y) \tag{3}$$

acting at the origin in the y–direction in a two-dimensional elastic solid with $\delta(x)$ being the Dirac delta function.

The solution of the problem (1)–(3) is well-known [6] and in polar coordinates reads:

$$\begin{aligned}\sigma_{rr}+\sigma_{\theta\theta} &= -\frac{Q_y}{2\pi(1-\nu)}\,\frac{\sin\theta}{r},\\ \sigma_{rr}-\sigma_{\theta\theta} &= -\frac{Q_y}{\pi}\,\frac{\sin\theta}{r},\\ \sigma_{r\theta} &= -\frac{(1-2\nu)Q_y}{4\pi(1-\nu)}\,\frac{\cos\theta}{r}.\end{aligned} \tag{4}$$

It is obvious that all the components of the stress tensor have non-physical singularities at the origin $r = 0$.

According to a nonlocal theory of continuum mechanics, the constitutive equations are obtained in such forms that the value of dependent constitutive variable at a point is described by the values of the independent variables at all points of the body. In particular, according to the nonlocal theory of elasticity, the stress tensor $\mathbf{t}$ at a reference point $\mathbf{x}$ in the body depends not only on the strains at this point but also on strains at all other points.

In this case the stress tensor is given by a weighted integral of the strains over the body

$$\mathbf{t}(\mathbf{x}) = \int_V \alpha(|\mathbf{x}'-\mathbf{x}|,\tau)\,\boldsymbol{\sigma}(\mathbf{x}')\,dV(\mathbf{x}'). \tag{5}$$

Here $\mathbf{x}$ and $\mathbf{x}'$ are reference and running points.

The weight function (the nonlocal modulus) $\alpha(|\mathbf{x}-\mathbf{x}'|,\tau)$ describes the nonlocal interaction, depends on a distance $|\mathbf{x}-\mathbf{x}'|$ between the reference $\mathbf{x}$ and running $\mathbf{x}'$ points, includes the parameter τ proportional to a characteristic length ratio a/l, where a is an internal characteristic length and l is an external characteristic length, is a delta–sequence, and in the the long–wavelength limit $\tau \to 0$ it becomes the Dirac delta–function

$$\lim_{\tau\to 0}\alpha(|\mathbf{x}-\mathbf{x}'|,\tau) = \delta(|\mathbf{x}-\mathbf{x}'|).$$

Eringen [7,8] has ascertained the properties of the nonlocal modulus and found several different forms giving a perfect match with the Born–Kármán model of the atomic lattice theory and the atomic dispersion curves. In the present paper we use the following two-dimensional nonlocal modulus

$$\alpha(|\mathbf{x}-\mathbf{x}'|,\tau)=\frac{1}{2\pi c^2}K_0\left(\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right), \tag{6}$$

where $K_0(x)$ is the modified Bessel function, $c = l\tau$.

As the modulus (6) is the fundamental solution of the Helmholtz equation, then action by the Helmholtz operator on two sides of the constitutive equation (5) it is possible to obtain the following inhomogeneous Helmholtz equation

$$c^2\Delta\mathbf{t}-\mathbf{t}=-\boldsymbol{\sigma}. \tag{7}$$

In polar coordinates the expressions for components of Laplacian of the symmetrical second order tensor have the following form [9]:

$$\begin{aligned}
(\Delta\mathbf{t})_{rr} &= \Delta\left(t_{rr}\right)-\frac{4}{r^2}\frac{\partial t_{r\theta}}{\partial\theta}-\frac{2}{r^2}\left(t_{rr}-t_{\theta\theta}\right),\\
(\Delta\mathbf{t})_{\theta\theta} &= \Delta\left(t_{\theta\theta}\right)+\frac{4}{r^2}\frac{\partial t_{r\theta}}{\partial\theta}+\frac{2}{r^2}\left(t_{rr}-t_{\theta\theta}\right),\\
(\Delta\mathbf{t})_{r\theta} &= \Delta\left(t_{r\theta}\right)-\frac{4}{r^2}t_{r\theta}+\frac{2}{r^2}\frac{\partial}{\theta}\left(t_{rr}-t_{\theta\theta}\right).
\end{aligned} \tag{8}$$

Let

$$t_{rr}=T_{rr}\sin\theta,\qquad t_{\theta\theta}=T_{\theta\theta}\sin\theta,\qquad t_{r\theta}=T_{r\theta}\cos\theta. \tag{9}$$

Then introducing three auxiliary functions

$$f(r)=T_{rr}+T_{\theta\theta},$$

$$g(r)=T_{rr}-T_{\theta\theta}+2T_{r\theta},$$
$$h(r)=T_{rr}-T_{\theta\theta}-2T_{r\theta}$$

and using expressions (8) we obtain from equation (7) the following inhomogeneous modified Bessel equations:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r}-\frac{1}{r^2}f-\frac{1}{c^2}f=\frac{Q_y}{2\pi(1-\nu)c^2}\frac{1}{r}, \tag{10}$$

$$\frac{\mathrm{d}^2 g}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}g}{\mathrm{d}r}-\frac{1}{r^2}g-\frac{1}{c^2}g=\frac{(3-4\nu)Q_y}{2\pi(1-\nu)c^2}\frac{1}{r}, \tag{11}$$

$$\frac{\mathrm{d}^2 h}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}h}{\mathrm{d}r}-\frac{9}{r^2}h-\frac{1}{c^2}h=\frac{Q_y}{2\pi(1-\nu)c^2}\frac{1}{r}. \quad (12)$$

The solutions of equations (10)–(12) are expressed in terms of modified Bessel functions $I_n\,(r/c)$ and $K_n\,(r/c)$.

As the function $I_n(r)$ is unbounded at infinity, the term with this function should be omitted. Using the formulae describing the behaviour of functions $K_n(r)$ for small values of r

$$K_1(r)\simeq\frac{1}{r},\qquad K_3(r)\simeq\frac{8}{r^3}-\frac{1}{r}$$

we arrive at the solution of equations (10)–(12) bounded both at infinity and at the origin

$$f(r)=\frac{Q_y}{2\pi(1-\nu)c}\left[K_1\left(\frac{r}{c}\right)-\frac{c}{r}\right], \quad (13)$$

$$g(r)=\frac{(3-4\nu)Q_y}{2\pi(1-\nu)c}\left[K_1\left(\frac{r}{c}\right)-\frac{c}{r}\right], \quad (14)$$

$$h(r)=-\frac{Q_y}{2\pi(1-\nu)c}\left[K_3\left(\frac{r}{c}\right)+\frac{c}{r}-8\left(\frac{r}{c}\right)^3\right]. \quad (15)$$

The solution of the problem obtained in this paper in the frame–work of nonlocal elasticity has no non-physical singularities.

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