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The Comparison of Semi-Orthogonal Wavelet and Orthonormal Wavelet Sets for Solving Fredholm's Integral Equation of the Second Kind

Jana Šimsová

Abstract. The article tries to resolve Fredholm's integral equations of the second kind by the Galerkin method. The wavelets on the bounded interval are used as the basic functions. Daubechies' orthonomal wavelets and the semi-orthogonal wavelets are used for solving Fredholm's integral equations of the second kind. The twofold results are evaluated in the function of the matrix-sparsity and the error of the approximation.

1 Introduction

Wavelets are very good tool for obtaining numerical approximations, because of their special properties as they are, for example, reproduced by polynomials up to the certain degree. In this section the essentials of the theory of wavelets are briefly summarized. The scaling function $\phi(x) \in L^2(R)$ satisfies dilation equation, namely

$$\phi(x) = \sqrt{2} \sum_{k \in Z} h_k \phi(2x - k).$$

The wavelet function $\psi \in L^2(R)$ satisfies wavelet identity

$$\psi(x) = \sqrt{2} \sum_{k \in Z} g_k \phi(2x - k).$$

Between scaling coefficients h_k and wavelet coefficients g_k there holds the relation $g_k = (-1)^{k-1} h_{1-k}$. The translations and dilatations of wavelet form an orthogonal base of space $L^2(R)$. We define space V_0 as $V_0 = span\{\phi(x - k)\}_{k \in Z}$. Scaling subspace V_j is generated by functions $\phi_{j,k}(x) = 2^{\frac{j}{2}} \phi(2^j x - k)$. The orthogonal complement of subspace V_j in V_{j+1} denoted W_j is generated by functions $\psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k)$. We say that the subspaces V_j generate multiresolution analysis.

The important property of wavelets is vanishing moment. A wavelet is said to have a vanishing moment of order m if

$$\int_{-\infty}^{\infty} x^p \psi(x)dx = 0, \quad p = 0, ..., m-1.$$

All wavelets satisfy the above condition for $p = 0$.

The wavelets $\{\psi_{j,k}\}$ form an orthonormal basis if

$$\langle \psi_{j,k}; \psi_{i,l} \rangle = \delta_{j,i}\delta_{k,l}, \qquad i,j,k,l, \in Z.$$

The wavelets $\{\psi_{j,k}\}$ form a semi-orthogonal basis if

$$\langle \psi_{j,k}; \psi_{i,l} \rangle = 0, \qquad i \neq j, \quad i,j,k,l, \in Z,$$

where $\langle .,. \rangle$ denote the scalar product in the space $L^2(R)$.

2 Orthonormal Wavelets on the Bounded Interval

In this section we present a very interesting approach to constructing orthonormal wavelet bases for $L^2_{\langle 0,1 \rangle}$. We start with family of Daubechie's wavelets on the real line. These wavelets have N zero moments, and scaling function ϕ and wavelet ψ are compactly supported on the interval $\langle -N+1, N \rangle$. So the width of both these functions is $2N-1$. It means that a number of scaling coefficients h_k in dilation equation and wavelet coefficients g_k, respectively, is finite. Without loss of generality we will consider only the interval $\langle 0; 1 \rangle$ instead of general bounded interval. It is easy to restrict the base of space $L^2(R)$ on the interval $\langle 0; 1 \rangle$. We can start from the base $\{\phi_{j_0,k} | k \in R\} \cup \{\psi_{j,k} | j \leq j_0; k \in Z\}$, where j_0 is chosen large enough so that support of functions $\phi_{j_0,k}$ and $\psi_{j_0,k}$ is included in the interval $\langle 0, 1 \rangle$. Therefore the least level j_0 must satisfy the inequality $2^{j_0-1} \geq N$. The number of these interior functions is equal to $2^j - 2N$. Now we add N left functions and N right functions, which are constructed on the right-half real line $\langle 0; +\infty)$ and on the left-half real line $(-\infty; 0\rangle$, respectively.

So scaling functions

$$\phi^*_{j,k}(x) = \begin{cases} \phi^{left}_{j,k}(x), & \text{if } \ 0 \leq k < N \\ \phi_{j,k}(x), & \text{if } \ N \leq k < 2^j - N \\ \phi^{right}_{j,k}(x), & \text{if } \ 2^j - N \leq k < 2^j \end{cases}$$

and wavelet functions

$$\psi^*_{j,k}(x) = \begin{cases} \psi^{left}_{j,k}(x), & \text{if } \ 0 \leq k < N \\ \psi_{j,k}(x), & \text{if } \ N \leq k < 2^j - N \\ \psi^{right}_{j,k}(x), & \text{if } \ 2^j - N \leq k < 2^j \end{cases}$$

generate an orthonormal multiresolution analysis on $\langle 0;1\rangle$. More details can be found in [1]. Furthermore, there exist the sets of coefficients $\{H_{k,l}^{left}\}, \{h_{k,l}^{left}\}, G_{k,l}^{left}\}, \{g_{k,l}^{left}\}, \{H_{k,l}^{right}\}, \{h_{k,l}^{right}\}, G_{k,l}^{right}\}, \{g_{k,l}^{right}\}$ such that the following recursive relation are valid:

$$\begin{cases} \phi_{j,k}^{left}(x) = \sum_{l=0}^{N-1} H_{k,l}^{left}\phi_{j+1,l}^{left}(x) + \sum_{l=N}^{N+2k} h_{k,l}^{left}\phi_{j+1,l}(x) \\ \psi_{j,k}^{left}(x) = \sum_{l=0}^{N-1} G_{k,l}^{left}\phi_{j+1,l}^{left}(x) + \sum_{l=N}^{N+2k} g_{k,l}^{left}\phi_{j+1,l}(x) \end{cases}$$

$$\begin{cases} \phi_{j,k}^{right}(x) = \sum_{l=-N}^{-1} H_{k,l}^{right}\phi_{j+1,l}^{right}(x) + \sum_{l=-N+1+2k}^{-N-1} h_{k,l}^{right}\phi_{j+1,l}(x) \\ \psi_{j,k}^{right}(x) = \sum_{l=-N}^{-1} G_{k,l}^{right}\phi_{j+1,l}^{left}(x) + \sum_{l=-N+1+2k}^{-N-1} g_{k,l}^{right}\phi_{j+1,l}(x) \end{cases}$$

Complete tables of these coefficients for Daubeschies wavelet and scaling function for N=2 are listed in [1]. These scaling functions and wavelets have not got explicit form. We must evaluate these functions by using either the cascade algorithm or recursive scheme.

3 Semi-orthogonal Wavelets on the Bounded Interval

Semi-orthogonal wavelet are derived from cardinal B-splines. B-splines N_m for $m \in N$ are recursively defined by integral convolution, namely $N_m(x) = \int_0^1 N_{m-1}(x-\tau)d\tau$, where $N_0(x) = \chi_{\langle 0;1\rangle}$ is the characteristic function of the interval $\langle 0;1\rangle$. This function is a suitable scaling function. There exist compactly supported wavelets with minimal support $\langle 0; 2m-1\rangle$ B-wavelets, given by $\psi_m(x) = \sum_k q_k N_m(2x-k)$, where nonzero coefficients q_k are only for $k = 0, \cdots, 3m-2$ given by

$$q_k = \frac{(-1)^k}{2^{m-1}} \sum_{l=0}^{m} \binom{m}{l} N_{2m}(k+1-l).$$

A base for $V_{j\langle 0;1\rangle}$ is given by B-spline of order m regarding the set of points $\{t_{-m+1}, t_{-m+2}, \ldots, t_{2^j+m-1}\}$, where $t_{-m+1} \cdots t_0 = 0$ and $t_{2^j} = \cdots = t_{2^j+m-1} = 1$ are knots of multiplicity m. The other knots are given by $t_k = k2^{-j}$. This base is defined as

$$B_{j,k}^m(x) = (t_{k+m} - t_k)[t_k, t_{k+1}, \ldots, t_{k+m}]_t (t-x)_+^{m-1},$$

where $[\cdot,\cdot,\cdot,\cdot]_t$ is the $m-th$ divided difference of $(t-x)_+^{m-1}$ with respect to the variable t. Therefore the inner scaling functions are dilatations and translations of the cardinal B-spline, namely $B^m_{j,k}(x) = N_m(2^jx - k)$ for $k = 0, \cdots, 2^j - m$. The inner wavelets $\psi_{j,k}$ for $k = 0, \cdots, 2^j - 2m + 1$ are given by

$$\psi_{j,k}(x) = \frac{1}{2^{2m-1}} \sum_{l=0}^{2m-2} (-1)^k N_{2m}(l+1) B^{2m}_{2k+l;j+1}(x).$$

The $m-1$ boundary wavelets for the endpoint 0 and the $m-1$ boundary wavelets for the endpoint 1 are linear combinations of the $m-1$ boundary splines and $2m+2k-1$ inner B-splines $N_{2m}(2^{j+1}x - k)$.

The second-order B-spline (scaling function) are given by

$$\phi_{j,k} = \begin{cases} 2^jx - k & x \in \langle \frac{k}{2^j}; \frac{k+1}{2^j} \rangle \\ 2 - (2^jx - k) & x \in \langle \frac{k+1}{2^j}; \frac{k+2}{2^j} \rangle \end{cases}$$

for

$$k = 0, 1, ..., 2^j - 2$$

with the respective left- and right-side boundary scaling functions

$$\phi_{j,-1} = 3 - 2^jx, \qquad x \in \langle 0, \frac{1}{2^j} \rangle,$$

$$\phi_{j,2^j-1} = 2^jx - 2^j + 1, \qquad x \in \langle \frac{2^j - 1}{2^j}, 1 \rangle$$

The second-order B-spline wavelets are given by

$$\psi_{j,k} = \frac{1}{6} \begin{cases} 2^jx - k & x \in \langle \frac{k}{2^j}; \frac{k+0.5}{2^j} \rangle \\ 4 - 7(2^jx - k) & x \in \langle \frac{k+0.5}{2^j}; \frac{k+1}{2^j} \rangle \\ -19 + 16(2^jx - k) & x \in \langle \frac{k+1}{2^j}; \frac{k+1.5}{2^j} \rangle \\ 29 - 16(2^jx - k) & x \in \langle \frac{k+1.5}{2^j}; \frac{k+2}{2^j} \rangle \\ -17 + 7(2^jx - k) & x \in \langle \frac{k+2}{2^j}; \frac{k+2.5}{2^j} \rangle \\ 3 - (2^jx - k) & x \in \langle \frac{k+2.5}{2^j}; \frac{k+3}{2^j} \rangle \end{cases}$$

for

$$k = 0, ..., 2^j - 3$$

with boundary wavelets

$$\psi_{j,k} = \frac{1}{6} \begin{cases} -6 + 23(2^jx) & x \in \langle 0; \frac{1}{2^{j+1}} \rangle \\ 14 - 17(2^jx) & x \in \langle \frac{1}{2^{j+1}}; \frac{1}{2^j} \rangle \\ -10 + 7(2^jx) & x \in \langle \frac{1}{2^j}; \frac{1.5}{2^j} \rangle \\ 2 - 2^jx & x \in \langle \frac{1.5}{2^j}; \frac{2}{2^j} \rangle \end{cases}$$

for the left-side boundary and

$$\psi_{j,k} = \frac{1}{6}\begin{cases} 2-(k+2-2^j x) & x \in \langle \frac{k}{2^j}; \frac{k+0.5}{2^j}\rangle \\ -10+7(k+2-2^j x) & x \in \langle \frac{k+0.5}{2^j}; \frac{k+1}{2^j}\rangle \\ 14-17(k+2-2^j x) & x \in \langle \frac{k+1}{2^j}; \frac{k+1.5}{2^j}\rangle \\ -6+23(k+2-2^j x) & x \in \langle \frac{k+1.5}{2^j}; 1\rangle \end{cases}$$

where $k = 2^j - 2$ for the right-side boundary.

Therefore, scaling functions and wavelets are piecewise linear functions.

4 Wavelet Expansion on $\langle 0;1\rangle$ and Integral Equation

We can approximate the function $f(x) \in L^2_{\langle 0;1\rangle}$ by its projection P_J on the space

$$V_J = span\{\phi^*_{J,k} | k = 0, 1, \ldots, 2^J - 1\},$$

where $2^J \geq 2N$ as

$$f(x) \approx P_{V_J} f(x) = \sum_{k=0}^{2^J-1} c_{J,k}\phi^*_{J,k}(x).$$

It follows from multiresolutions that we can write

$$P_{V_J} f(x) = \sum_{k=0}^{2^{j_0}-1} c_{j_0,k}\phi^*_{j_0,k}(x) + \sum_{j=j_0}^{J}\sum_{k=0}^{2^j-1} d_{j,k}\psi^*_{j,k}(x).$$

So we use small details at levels j_0 to J and the coarsest details at a level j_0. Now we consider Fredholm's integral equation of the second kind in the form

$$f(x) + \int_0^1 K(x,y) f(y) dy = g(x), \tag{1}$$

where $g(x)$ is a known function, $K(x,y)$ is a kernel of integral equation and $f(x)$ is an unknown function from $L^2_{\langle 0,1\rangle}$.

Applying the wavelet expansion of projection P_{V_J} of unknown function f into integral equation and scalar multiplying by family of functions $\{\phi^*_{j_0,k}\}_{k=2}^{2^{j_0}-1} \cup \{\psi^*_{j,k}\}_{j=j_0;k=0}^{J;2^j-1}$ we obtain the following system of linear equations:

$$\left\{\begin{bmatrix} X_{\phi,\phi} & X_{\phi,\psi} \\ X_{\psi,\phi} & X_{\psi,\psi}\end{bmatrix} + \begin{bmatrix} Y_{\phi,\phi} & Y_{\phi,\psi} \\ Y_{\psi,\phi} & Y_{\psi,\psi}\end{bmatrix}\right\} * \begin{bmatrix} c_{j_0,k} \\ d_{j,k}\end{bmatrix} = \begin{bmatrix} G_\phi \\ G_\psi \end{bmatrix}$$

where the elements of matrices X and Y are

$$X_{\phi,\phi}=\int_0^1 \phi_{i_0,l}(x)\phi_{j_0,k}(x)dx,$$

$$X_{\phi,\psi}=\int_0^1 \phi_{i_0,l}(x)\psi_{j,k'}(x)dx,$$

$$X_{\psi,\phi}=\int_0^1 \psi_{i,l'}(x)\phi_{j_0,k}(x)dx,$$

$$X_{\psi,\psi}=\int_0^1 \psi_{i,l'}(x)\psi_{j,k'}(x)dx;$$

$$Y_{\phi,\phi}=\int_0^1\int_0^1 \phi_{i_0,l}(x)\phi_{j_0,k}(y)K(x,y)dydx,$$

$$Y_{\phi,\psi}=\int_0^1\int_0^1 \phi_{i_0,l}(x)\psi_{j,k'}(y)K(x,y)dydx,$$

$$Y_{\psi,\phi}=\int_0^1\int_0^1 \psi_{i,l'}(x)\phi_{j_0,k}(y)K(x,y)dydx,$$

$$Y_{\psi,\psi}=\int_0^1\int_0^1 \psi_{i,l'}(x)\psi_{j,k'}(y)K(x,y)dydx,$$

and the subscripts k,l,k',l',i,j, are given as $k,l=-1,...,2^{j_0}-1; k',l'=-1,...,2^j-1$ and $j_0\le j\le J; i_0\le i\le I$ for semi-orthogonal wavelets and $k,l=0,.....,2^{j_0}-1; k',l'=0,......2^j-1$ and $j_0\le j\le J; i_0\le i\le I$ for orthonormal Daubechies wavelets. The elements of vector G_ϕ are integrals $\int_0^1 g(x)\phi_{i_0,l}(x)dx$ and the elements G_ψ are integrals $\int_0^1 g(x)\psi_{i,l}(x)dx$.

5 Numerical Example

In this section the use of introduced scaling functions and wavelets is compared by numerical example. The considered integral equation is

$$f(x)+\int_0^1 (x+y-2xy)f(y)dy=x^2+x.$$

The kernel of integral equation has no singularity. A total number of unknowns N in this system of linear equations is $N=2^{J+1}+1$, where J is the highest level of decomposition, for semi-orthogonal wavelets and total number of unknowns $N=2^{j_0+3}$, where j_0 is the smallest level of decomposition, for orthonormal wavelets. In both bases there is the smallest level $j_0=2$ and the highest level $J=4$. Therefore $N=33$ when the semi-orthogonal wavelets were used and $N=32$ when the orthogonal wavelets were used.

If we remember the properties of SO-wavelets, we know that the matrix $X_{\phi,\phi}$ is a diagonal matrix, $X_{\phi,\psi}$ and $X_{\psi,\phi}$ are zero matrices and $X_{\psi,\psi}$ is the block-five diagonal matrix. In a case of orthonormal D-wavelets the matrix X is the identity matrix. Look in detail on the elements of matrix Y. Even though the limits of integration in every element of matrix Y range from zero to one, the actual integration limits are much smaller because of the finite supports of both using bases. The matrix $Y_{\phi,\phi}$ is a dense matrix with not very small elements. But this matrix occupies very small (5×5 for SO-scaling function and 4×4 for orthonormal scaling function) part of the matrix Y. The matrices $Y_{\phi,\psi}; Y_{\psi,\phi}$ and $Y_{\psi,\psi}$ are dense too. But because of local supports and vanishing moment properties of wavelets many elements of these matrices are very small compared to the largest element. Hence, they can be dropped without significantly influence on the solution. So, the elements whose magnitudes are smaller then $\epsilon = 0.001$ can be set as zero (ϵ is called a threshold parameter). In figure 1 and figure 2 we can see nonzero elements of matrix $X+Y$ as dark dots. We can see that orthonormal base gives a sparser matrix.

If we want to evaluate the elements of matrix Y, we need not compute the double integrals in every element of matrix. It is enough to use any numerical quadrature only on the high level and then work only with sets of scaling and wavelets coefficients. In this numerical example the trapezoid rule was used.

The relative quadratic error in the form $\epsilon = \sum_{i=1}^{n} (\overline{f}(x_i) - f(x_i))^2 / f(x_i)^2$ (where $\overline{f}$ is an approximation and f is an exact solution) has the values 2.99×10^{-5} in the case of SO-wavelets and 5.49×10^{-3} in the case of orthonormal wavelets. So SO-wavelets better approximate the exact solution. The advantage of SO-wavelets is their explicit form, piecewise linear behavior (for m=2) and their symmetry graph.

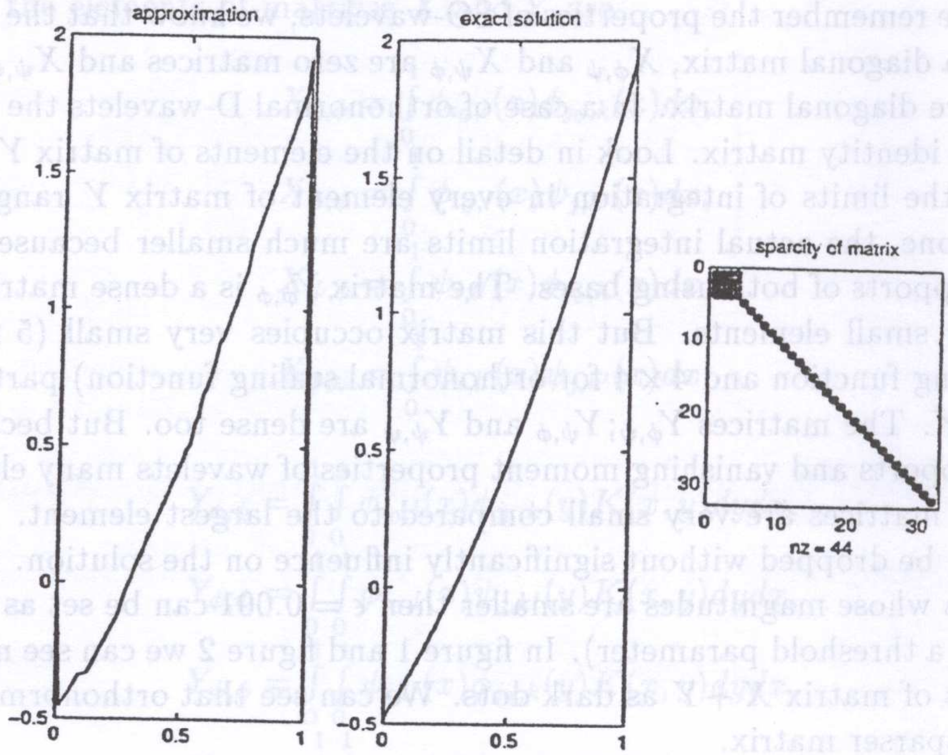

Fig. 1 Approximation by Daubechies-orthogonal wavelets.

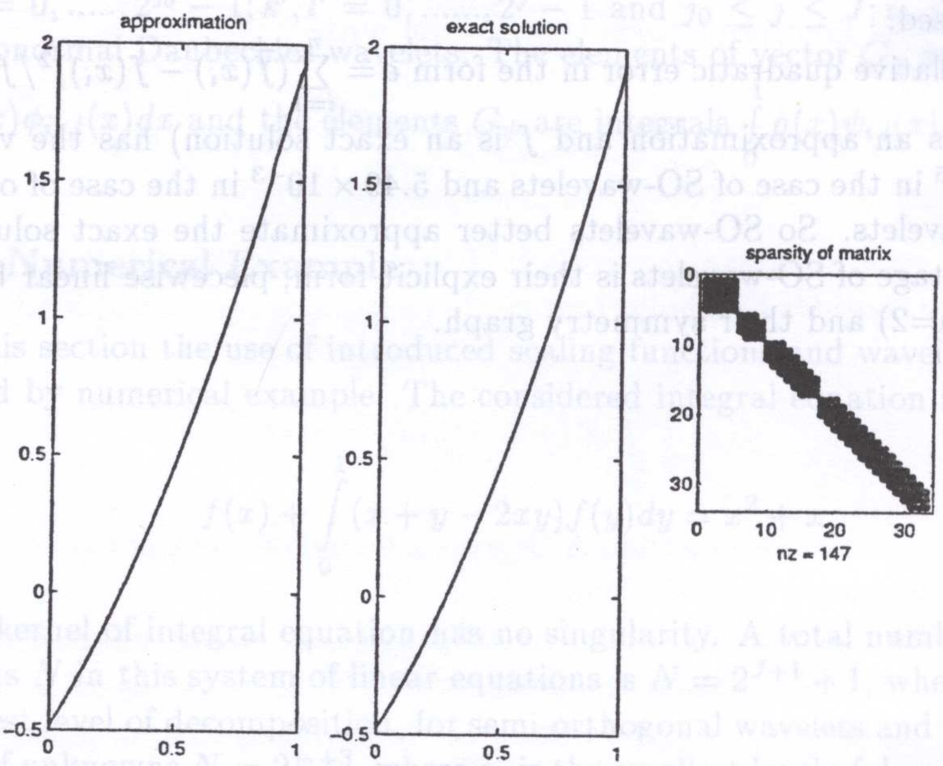

Fig. 2 Approximation by semi-orthogonal wavelets on the interval $\langle 0; 1 \rangle$.

References

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J.E.Purkyně University
České mládeže 8
400 96 Ústí nad Labem
Czech Republic