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Two Priority Head-of-the-line Queueing System with Random Length Demands

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We consider two-priority head-of-the-line queueing system $\mathbf{M_2/G_2/1/\infty}$. Let a_i be an intensity of i-th priority demands entrance flow, $i = 1, 2$. Denote as $a = a_1 + a_2$ the intensity of summarized entrance flow. Each demand of i-th priority is characterized by some random length (volume) ζ_i, not depending neither on other demands volumes nor on arrival time moment of the demand [1]. Suppose, that the demand volume ζ_i and service time ξ_i can be dependent. The joint distribution of ζ_i and ξ_i random variables is described by the following distribution function: $F_{(i)}(x,t) = \mathbf{P}\{\zeta_i < x, \xi_i < t\}$. Then the distribution functions of ζ_i and ξ_i random variables are equal $L_{(i)}(x) = F_{(i)}(x,\infty)$, $B_{(i)}(t) = F_{(i)}(\infty,t)$ correspondingly. Denote as $\alpha_{(i)}(s,q)$ the double Laplace-Stieltjes transform (LST) of $F_{(i)}(x,t)$ distribution function. The mixed $j+k$-th moments (if they exist) of two-dimensional random vector (ζ_i, ξ_i) we shall denote as $\alpha_{(i)jk}$, $j,k = 1,2,\ldots$

Denote as $\varphi_{(i)}(s) = \alpha_{(i)}(s,0)$, $\beta_{(i)}(q) = \alpha_{(i)}(0,q)$ LST of distribution functions $L_{(i)}(x)$ and $B_{(i)}(t)$ correspondingly. The j-th moments of ζ_i and ξ_i random variables (if they exist) denote as $\varphi_{(i)j}$ and $\beta_{(i)j}$. Let $\sigma_{(i)}(t)$ be the total sum of volumes of i-th priority demands presenting in the system at time moment t. Thus, the system is characterized by random vector $\sigma(t) = (\sigma_{(1)}(t), \sigma_{(2)}(t))$ at arbitrary time moment t. Suppose that random variables $\sigma_{(1)}(t)$ and $\sigma_{(2)}(t)$ are not restricted.

We assume that in the system under consideration the server is unreliable in a free state in the following sense. If at some time moment T the server becomes free and stay free (i.e. there are no other demands in the system at this moment) and demands doesn't come to the system after this moment during time t, then the server will disturb in time interval $[T; T+t)$ with the probability $E(t)$. After that the server will rehabilitate oneself during random time, which is characterized by distribution function $H(t)$.

Demands coming to the system during server rehabilitation wait in the queue for termination of this process, then they will be served in accordance to head-of-the-line priority discipline. Denote as $\varepsilon(q)$ and $h(q)$ LST of $E(t)$ and $H(t)$ functions accordingly. Let ε_j and h_j be the j-th moments of correspondent random variables.

Let $D(x_1, x_2, t)$ be the distribution function of $\boldsymbol{\sigma}(t)$ random vector: $D(x_1, x_2, t) = \mathbf{P}\{\sigma_{(1)}(t) < x_1, \sigma_{(2)}(t) < x_2\}$.

Denote as $\overline{\delta}(s_1, s_2, t)$ the double (on x_1 and x_2) LST of $D(x_1, x_2, t)$ distribution function, and as $\delta(s_1, s_2, q)$ the Laplace transform with respect to t of $\overline{\delta}(s_1, s_2, t)$ function. Our aim is determination of $\delta(s_1, s_2, q)$ function. In spite of this we assume that at the time moment $t = 0$ there were no demands in the system and the server was in reliable state. The problem under consideration is the generalization of demands number determination one, which was formulated in [2] for so-called two-priority head-of-the-line system of the first type.

It's known that the stationary mode exists for the system under consideration, if $a_1\beta_{(1)1} + a_2\beta_{(2)1} < 1$. In this case the limit $D(x_1, x_2) = \lim\limits_{t\to\infty} D(x_1, x_2, t)$ exists, where $D(x_1, x_2) = \mathbf{P}\{\sigma_1 < x_1, \sigma_2 < x_2\}$ is the joint distribution function of stationary total volumes σ_1, σ_2 of the first and second priority demands accordingly.

Let $\delta(s_1, s_2)$ be the double LST of $D(x_1, x_2)$ distribution function. It follows from renewal process theory [3] that

$$\delta(s_1, s_2) = \lim_{t\to\infty} \overline{\delta}(s_1, s_2, t) = \lim_{q\downarrow 0} q\delta(s_1, s_2, q).$$

For $\delta(s_1, s_2, q)$ function determination we use the modified additional variables method [4,5], which differs from the conventional one [6], so that demand colouring is dependent of its volume, i.e. the demand of i-th priority, $i = 1, 2$, having the volume x, will be red with probability $e^{-s_i x}(s_i > 0)$ or blue with probability $1 - e^{-s_i x}$.

So, the correspondent random variables transforms obtain the following probability sense: $\varphi_{(i)}(s_i)$ is the probability that an arbitrary demand of i-th priority will is red; $\beta_{(i)}(q)$ is the probability that there will be no catastrophes during service time of the demand; $\alpha_{(i)}(s_i, q)$ is the joint probability that an arbitrary i-th priority demand will be red and there will be no catastrophes during service time of the demand; $\overline{\delta}(s_1, s_2, t)$ is the probability that there are no blue demands in the system at the time moment t; $q\delta(s_1, s_2, q)$ is the probability that at the moment of the first catastrophe coming there were no blue demands in the system; $\delta(s_1, s_2)$ is the probability that there were no blue demands in the system in stationary mode.

It follows from [1] that the conditional probability, that i-th priority demand being on service is red under condition, that time y passed from the service beginning, is equal to

$$e_{y\,(i)}(s_i) = \left[1 - B_{(i)}(y)\right]^{-1} R_{(i)}(s_i, y),$$

where

$$R_{(i)}(s_i, y) = \int_{x=0}^{\infty} e^{-s_i x} \int_{u=y}^{\infty} dF_{(i)}(x, u).$$

By modified additional variables method we obtain the following statement.

Theorem. *For the system under consideration the function $\delta(s_1, s_2, q)$ can be represented in the following form:*

$$\delta(s_1, s_2, q) = \frac{1}{\mu(q)} \left\{ \frac{1 - \varepsilon(a+q)}{a+q} + \frac{\varepsilon(a+q)[1 - h(\kappa(q, s_1, s_2))]}{\kappa(q, s_1, s_2)} + \right.$$

$$+ B(q, s_1, s_2) \frac{\varphi_{(1)}(s_1) - \alpha_{(1)}(s_1, \kappa(q, s_1, s_2))}{\kappa(q, s_1, s_2)} +$$

$$\left. + C(q, s_2) \frac{\varphi_{(2)}(s_2) - \alpha_{(2)}(s_2, \kappa(q, s_1, s_2))}{\kappa(q, s_1, s_2)} \right\},$$

where

$$\mu(q) = 1 - \varepsilon(a+q) h(a + q - a\pi(q)) - \frac{a}{a+q} [1 - \varepsilon(a+q)] \pi(q),$$

$\pi(q)$ is the LST of busy period of the system (i.e. the time period from demand entrance to reliable free system to the nearest moment when the system will be reliable and free again),

$$\kappa(q, s_1, s_2) = q + a_1(1 - \varphi_{(1)}(s_1)) + a_2(1 - \varphi_{(2)}(s_2)),$$

$$B(q, s_1, s_2) = \frac{\varepsilon(a+q)}{\varphi_{(1)}(s_1) - \beta_{(1)}(\kappa(q, s_1, s_2))} \left\{ h(\kappa(q, s_1, s_2)) - h(\chi(q, s_2)) + \right.$$

$$\left. + \frac{[\beta_{(2)}(\kappa(q, s_1, s_2)) - \beta_{(2)}(\chi(q, s_2))][h(\chi(q, s_2)) - h(\psi(q))]}{\varphi_{(2)}(s_2) - \beta_{(2)}(\chi(q, s_2))} \right\} +$$

$$+ \frac{1 - \varepsilon(a+q)}{(a+q)(\varphi_{(1)}(s_1) - \beta_{(1)}(\kappa(q, s_1, s_2)))} \times$$

$$\times \left\{ a_1[\varphi_{(1)}(s_1) - \pi_{(1)}(q + a_2(1 - \varphi_{(2)}(s_2)))] + \frac{\beta_{(2)}(\kappa(q, s_1, s_2)) - \beta_{(2)}(\chi(q, s_2))}{\varphi_{(2)}(s_2) - \beta_{(2)}(\chi(q, s_2))} \times \right.$$

$$\times\Big[a_1\Big(\pi_{(1)}(q+a_2(1-\varphi_{(2)}(2)))-\pi_{(1)}(q+a_2(1-\pi^*_{(2)}(q)))\Big)+ +a_2(\varphi_2(s_2)-\pi^*_{(2)}(q))\Big]\Big\},$$

$$\chi(q,s_2)=q+a_1[1-\pi_{(1)}(q+a_2(1-\varphi_{(2)}(s_2)))]+a_2(1-\varphi_{(2)}(s_2)),$$

$\pi_{(1)}(q)$ is LST of busy period of $M/G/1/\infty$ system with a_1 entrance flow intensity and the distribution function of service time $B_{(1)}(t)$, $\pi^*_{(2)}(q)$ is determined from the equation $\pi^*_{(2)}(q)=t(q+a_2-a_2\pi^*_{(2)}(q))$, $\mathrm{Re}\, q>0$, $|\pi^*_{(2)}(q)|<1$, where $t(q)=\beta_{(2)}(q+a_1(1-\pi_{(1)}(q)))$,

$$C(q,s_2)=\frac{1}{\varphi_{(2)}(s_2)-\beta_{(2)}(\chi(q,s_2))}\Big\{\varepsilon(a+q)[h(\chi(q,s_2))-h(\psi(q))]+ +\frac{1-\varepsilon(a+q)}{(a+q)}\Big[a_1(\pi_{(1)}(q+a_2(1-\varphi_{(2)}(s_2)))- -\pi_{(1)}(q+a_2(1-\pi^*_{(2)}(q)))+a_2(\varphi_{(2)}(s_2)-\pi^*_{(2)}(q))\Big]\Big\},$$

$$\psi(q)=q+a_1[1-\pi_{(1)}(q+a_2(1-\pi^*_{(2)}(q)))]+a_2(1-\pi^*_{(2)}(q)).$$

In stationary mode we obtain the next expression for the function $\delta(s_1,s_2)$:

$$\delta(s_1,s_2)=\frac{a(1+a\beta_1)}{1-\varepsilon(a)(1-ah_1)}\Big\{\frac{1-\varepsilon(a)}{a}+\frac{\varepsilon(a)[1-h(\kappa(s_1,s_2))]}{\kappa(s_1,s_2)}+ +B(s_1,s_2)\frac{\varphi_{(1)}(s_1)-\alpha_{(1)}(s_1,\kappa(s_1,s_2))}{\kappa(s_1,s_2)}+ +C(s_2)\frac{\varphi_{(2)}(s_2)-\alpha_{(2)}(s_2,\kappa(s_1,s_2))}{\kappa(s_1,s_2)}\Big\},$$

where $\beta_1=\frac{a_1}{a}\beta_{(1)1}+\frac{a_2}{a}\beta_{(2)1}$, $\kappa(s_1,s_2)=a_1(1-\varphi_{(1)}(s_1))+a_2(1-\pi_{(2)}(s_2))$, $\chi(s_2)=a_1\{1-\pi_{(1)}(a_2-a_2\varphi_{(2)}(s_2))\}+a_2-a_2\varphi_{(2)}(s_2)$,

$$B(s_1,s_2)=\frac{\varepsilon(a)}{\varphi_{(1)}(s_1)-\beta_{(1)}(\kappa(s_1,s_2))}\Big\{h(\kappa(s_1,s_2))-h(\chi(s_2))+ +\frac{h(\chi(s_2))-1}{\varphi_{(2)}(s_2)-\beta_{(2)}(\chi(s_2))}\Big[\beta_{(2)}(\kappa(s_1,s_2))-\beta_{(2)}(\chi(s_2))\Big]\Big\}+ +\frac{1-\varepsilon(a)}{a(\varphi_{(1)}(s_1)-\beta_{(1)}(\kappa(s_1,s_2)))}\Big\{a_1\Big[\varphi_{(1)}(s_1)-\pi_{(1)}(a_2(1-\varphi_{(2)}(s_2)))\Big]+$$

$$+\frac{\chi(s_2)[\beta_{(2)}(\chi(s_2))-\beta_{(2)}(\kappa(s_1,s_2))]}{\varphi_{(2)}(s_2)-\beta_{(2)}(\chi(s_2))}\bigg\},$$

$$C(s_2)=\frac{a\varepsilon(a)[h(\chi(s_2))-1]-(1-\varepsilon(a))\chi(s_2)}{a[\varphi_{(2)}(s_2)-\beta_{(2)}(\chi(s_2))]}.$$

Now we can easy obtain the formulae for LST $\delta_{(1)}(s)=\delta(s,0)$ and $\delta_{(2)}(s)=\delta(0,s)$ of stationary total volume for demands of each priority. Then we can calculate the first stationary moments of each priority total demands volume. The result of this calculation for demands of the first and second priority accordingly is

$$\delta_{(1)1}=a_1\alpha_{(1)11}+\frac{aa_1\varphi_{(1)1}}{2\left(1-a_1\beta_{(1)1}\right)}\left[\beta_2+\frac{h_2\varepsilon(a)\,(1-a\beta_1)}{1-\varepsilon(a)(1-ah_1)}\right],$$

$$\delta_{(2)1}=a_2\alpha_{(2)11}+\frac{aa_2\varphi_{(2)1}}{2\left(1-a_1\beta_{(1)1}\right)}\left[\frac{\beta_2}{1-a\beta_1}+\frac{h_2\varepsilon(a)}{1-\varepsilon(a)\,(1-ah_1)}\right],$$

where $\beta_2=\frac{a_1}{a}\beta_{(1)2}+\frac{a_2}{a}\beta_{(2)2}$.

From the last two formulae we obtain the following expression for the first stationary moment of the total volume of demands presenting in the system:

$$\delta_1=\delta_{(1)1}+\delta_{(2)1}=$$

$$=a\alpha_{11}+\frac{a^2\left(\varphi_1-a_1\beta_1\varphi_{(1)1}\right)}{2\left(1-a_1\beta_{(1)1}\right)}\left[\frac{\beta_2}{1-a\beta_1}+\frac{h_2\varepsilon(a)}{1-\varepsilon(a)\,(1-ah_1)}\right],$$

where $\alpha_{11}=\frac{a_1}{a}\alpha_{(1)11}+\frac{a_2}{a}\alpha_{(2)11}$, $\varphi_1=\frac{a_1}{a}\varphi_{(1)1}+\frac{a_2}{a}\varphi_{(2)1}$.

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