Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka IX, Częstochowa 2003

Algorithms for Composing Magic Cubes

Marián Trenkler

Abstract:

A *magic square* (a square array containing natural numbers $1, 2, \ldots, n^2$ such that the sum of the numbers along every row, column and diagonal is the same) has fascinated people for centuries. In 1686, the Polish mathematician *Adamas Kochansky* extended magic squares to three dimensions. An additive magic cube is a natural generalization of a magic square.

An *additive magic cube* of order n is a cubical array

$$\mathbf{M}_n = |\mathbf{m}_n(i,j,k); \quad 1 \le i,j,k \le n|$$

containing natural numbers $1, 2, 3, \ldots, n^3$ such that the sum of the numbers along every row and great diagonal is the same, i.e. $\frac{n(n^3+1)}{2}$. By a row of a magic cube we mean an n-tuple of elements having the same coordinates on two places. Every additive magic cube of order n has exactly $3n^2$ rows and 4 diagonals. Figure 1 depicts an additive magic cube $\mathbf{M}_3$. The element $\mathbf{m}_3(1,1,1) = 8$ is contained in the rows $\{8, 15, 19\}$, $\{8, 24, 10\}$, $\{8, 12, 22\}$ and on the diagonal $\{8, 14, 20\}$.

8	15	19
24	1	17
10	26	6

1st layer

12	25	5
7	14	21
23	3	16

2nd layer

22	2	18
11	27	4
9	13	20

3th layer

Figure 1.

A *multiplicative magic cube* of order n is a cubical array

$$\mathbf{Q}_n = |\mathbf{q}_n(i,j,k); \ 1 \le i,j,k \le n|$$

containing n^3 mutually different natural numbers such that the product of the numbers along each row and every of its four diagonals is the same. We call this product *magic constant* and denote $\sigma(\mathbf{Q}_n)$.

In [4] it is proved that an additive magic cube $\mathbf{M}_n$ of order n exists for every $n \neq 2$. If we know a construction of $\mathbf{M}_n = |\mathbf{m}_n(i,j,k)|$, then we can easily make a multiplicative magic cube

$$\mathbf{Q}_n = |\mathbf{q}_n(i,j,k) = 2^{\mathbf{m}_n(i,j,k)-1}| \tag{1}$$

with the magic constant $\sigma(\mathbf{Q}_n) = 2^{\frac{n(n^3-1)}{2}}$.

This paper contains formulas for construction of magic cubes $\mathbf{M}_n$ and $\mathbf{Q}_n$ for all $n \neq 2$. Moreover the constructed cubes $\mathbf{Q}_n$ have a significantly smaller magic constant than cubes constructed using (1). The correctness of our formulas follows immediately from the proofs in [4,5].

We construct an additive magic cube $\mathbf{M}_n = |\mathbf{m}_n(i,j,k)|$ of order n and a multiplicative magic cube $\mathbf{Q}_n = |\mathbf{q}_n(i,j,k)|$ of order n for all $n \neq 2$ using the following formulas. We consider three cases and we use the following notation: $\overline{x} = n+1-x$, $x^* = \min\{x, \overline{x}\}$, $\tilde{x} = 0$ for $1 \leq x \leq \frac{n}{2}$ and $\tilde{x} = 1$ for $\frac{n}{2} < x \leq n$.

1. If $n \equiv 1 (\mathrm{mod}\ 2)$ then

$$\mathbf{m}_n(i,j,k) = \alpha n^2 + \beta n + \gamma + 1, \tag{2}$$

$$\mathbf{q}_n(i,j,k) = 2^{\alpha}.3^{\beta}.5^{\gamma}, \tag{3}$$

where $\alpha = (i - j + k - 1) \pmod{n}$, $\beta = (i - j - k) \pmod{n}$, $\gamma = (i + j + k - 2) \pmod{n}$.

Note: If $n \not\equiv 0 \pmod 3$ then not only in every row but also on every diagonal $\mathbf{Q}_n$ constructed by 1 there is exactly one number which is divisible by the z-th power but is not divisible by the $(z+1)$-th power of the number 2 (3 or 5, respectively). We obtain a multiplicative magic cube $\mathbf{Q}_n$ with a smaller magic constant $\sigma(\mathbf{Q_n})$ if in the formula (3) we replace 3^{β} by the number $(2\beta + 1)$ for all $\beta = 1, 2, \ldots, n-1$ and 5^{γ} by the number $(2n + 2\gamma - 1)$ for $\gamma = 1, 2, \ldots, n-1$. (See [6].)

2. If $n \equiv 0 (\mathrm{mod}\ 4)$, then

$$\mathbf{m}_n(i,j,k) = \begin{cases} (i-1)\, n^2 + (j-1)\, n + k & \text{if} \quad \mathcal{F}(i,j,k) = 1 \\ (\overline{i}-1)\, n^2 + (\overline{j}-1)\, n + \overline{k} & \text{if} \quad \mathcal{F}(i,j,k) = 0 \end{cases} \tag{4}$$

$$\mathbf{q}_n(i,j,k) = \begin{cases} 2^{(i-1)}.3^{(j-1)}.5^{(k-1)} & \text{if} \quad \mathcal{F}(i,j,k) = 1 \\ 2^{(\overline{i}-1)}.3^{(\overline{j}-1)}.5^{(\overline{k}-1)} & \text{if} \quad \mathcal{F}(i,j,k) = 0 \end{cases} \tag{5}$$

where $\mathcal{F}(i,j,k) = (i + \tilde{i} + j + \tilde{j} + k + \tilde{k}) \pmod 2$.

Note: Using another construction we can make a cube $\mathbf{Q}_n$ with a smaller magic constant $\sigma(\mathbf{Q}_n)$. We demonstrate this construction on the following example. Figure 2 depicts four layers of $\mathbf{M}_4$ (constructed by (4)) whose numbers are the binary representation of the numbers $\mathbf{m}_4(i,j,k)-1$.

000000	111110	111101	000011
111011	000101	000110	111000
110111	001001	001010	110100
001100	110010	110001	001111

$1st$ layer

101111	010001	010010	101100
010100	101010	011001	010111
011000	100110	100101	011011
100011	011101	011110	100000

$2nd$ layer

011111	100001	100010	011100
100100	011010	011001	100111
101000	010110	010101	101011
010011	101101	101110	010000

$3th$ layer

110000	001110	001101	110011
001011	110101	110110	001000
000111	111001	111010	000100
111100	000010	000001	111111

$4th$ layer

Figure 2.

By closely examining Figure 2 you can find out that in every 4-tuple of numbers in any row or diagonal it holds that on the z-th position, $z = 1, 2, \dots, 6$, there are exactly two ones and two zeroes. We use this fact in the construction. If $b_1b_2b_3\dots b_6$ is the representation of the number $\mathbf{m}_4(i,j,k)$ lowered by 1 in binary code then

$$q_4(i,j,k) = 2^{b_1}3^{b_2}4^{b_3}5^{b_4}7^{b_5}9^{b_6}.$$

We have chosen the set $\{2,3,4,5,7,9\}$ in such a way that it does not contain two nonempty subsets of numbers whose product is the same. The magic constant of $\mathbf{Q}_4$ (on Figure 3) is $\sigma(\mathbf{Q}_4) = (2.3.4.5.7.9)^2 = 57\ 153\ 600$.

1	840	1080	63
1512	45	35	24
1890	36	28	30
20	42	54	1260

$1st$ layer

2520	27	21	40
15	56	72	945
12	70	90	756
126	540	420	2

$2nd$ layer

3780	18	14	60
10	84	108	630
8	105	135	504
189	360	280	3

3th layer

6	140	180	378
252	270	210	4
315	216	168	5
120	7	9	7560

4th layer

Figure 3.

3. If $n \equiv 2 \pmod 4$ (in this case $m = \frac{n}{2}$ is odd), then

$$\mathbf{m}_n(i,j,k) = s(u,v)\frac{n^3}{8} + \mathbf{m}_m(i^*, j^*, k^*),$$

$$\mathbf{q}_n(i,j,k) = 7^{s(u,v)} * \mathbf{q}_m(i^*, j^*, k^*),$$

where

$\mathbf{m}_m$ and $\mathbf{q}_m$ are constructed by (2) and (3),

$u = (i^* - j^* + k^*) \pmod m + 1,$

$v = 4\tilde{i} + \tilde{j} + \tilde{k},$

$s(u,v)$ for $1 \le u \le m, 1 \le v \le 8$ is defined by the following table

$(a = 1, 2, \ldots, \dfrac{n-6}{4})$

	s(u, 1)	s(u, 2)	s(u, 3)	s(u, 4)	s(u, 5)	s(u, 6)	s(u, 7)	s(u, 8)
s(1, v)	7	3	6	2	5	1	4	0
s(2, v)	3	7	2	6	1	5	0	4
s(3, v)	0	1	3	2	5	4	6	7
s(2a + 2, v)	0	1	2	3	4	5	6	7
s(2a + 3, v)	7	6	5	4	3	2	1	0

Remark. By the end of the 19-th century mathematicians began to consider also 4-dimensional magic cubes. But only in 2001 the following theorem was published: *An additive magic d-dimensional cube of order n exists if and only if $d > 1$ and $n \neq 2$ or $d = 1$.* Similarly we can consider the existence of multiplicative magic d-dimensional cubes for any natural d.

References

[1] W.S.Andrews, *Magic Squares and Cubes*, Dover, New York, 1960.

[2] D.Borkovitz, F.K.Hwang, *Multiplicative magic squares*, Discrete Mathematics **47**, 1–11, 1983.

[3] I.Semanišinová, *Objavovanie čara magických štvorcov*, Disputationes Scientificae Universitatis Catholicae in Ružomberok 4, 86–92, 2002.

[4] M.Trenkler, *A construction of magic cubes*, The Mathematical Gazette **84**, 36–41, 2000.

[5] M.Trenkler, *Magic p-dimensional cubes*, Acta Arithmetica **96**, 361–364, 2001.

[6] M.Trenkler, *Súčinové magické štvorce a kocky*, Obzory matematiky, fyziky a informatiky **1**, 9–16, 2002.

Department of Mathematics and Physics
Catholic University in Ružomberok
Place A.Hlinka 56, 034 01 Ružomberok,
Slovak Republic
e-mail: trenkler@science.upjs.sk