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An algebraic characterization of some reducts of three–valued logics

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Introduction. The aim of this paper is to present certain class of logics over finite algebras, in particular over three–valued algebras. The logics certain properties of those algebras having an automorphism, congruences or subalgebras, reduce to logics over finite algebras with the universe including fewer elements. So that some three–valued logics. In the paper [7] the method of axiomatizing the logic arbitrary two–valued algebra has been presented, provided that the set of tautologies of this logic is non – empty.

The axiomatization of the logic over three–valued algebras, particular over the reducts of these algebras may, be reduced in some cases to the problem of two–valued logics.

1. Preliminaries.

Let N be the set of nonnegative integers, let $V = \{p_1 : i \in N\}$ be a denumerable set of propositional variables and let F be am $m - ary$ logical connective, where m is a fixed natural number ≥ 2. S is the set defined by inductively in the following way:

(1) $p_i \in S$ for all $i \in N$

(2) if $\alpha_1, \alpha_2, \ldots, \alpha_m$ are elements of S then
$F(\alpha_1, \alpha_2, \ldots, \alpha_m)$ is an element of S.

(3) S is the least set safisfying (1) and (2).

The elements of S are called formulae over V and F. The pair $\mathcal{S} = (S, F)$ is called the language over V and F.

A logical matrix for the language $\mathcal{S}$ is any triple $\mathcal{E} = (E, E^*, f)$, where $E = \{1, 2, \ldots, n\}$ for some natural number $n \geq 2$ $E^* = \{r, r+1, \ldots, n\}$ for some natural number $r, 1 < r \leq n$, and $f : E^m \to E$ is a function.

For any function e_v defined by inductively by:

(1) $e_v(p) = v(p)$

(2) $e_v(F(\alpha_1, \alpha_2, \ldots, \alpha_m)) = f(e_v(\alpha_1), e_v(\alpha_2), \ldots, e_v(\alpha_m))$ for

$\alpha_1, \alpha_2, \ldots, \alpha_m \in S$,

instead of $e_v(\alpha)$ we will also write $\|\alpha\|(v)$. The formula α is a *tautology* over $\mathcal{E}$ if $e_v(\alpha) \in E^+$ for every v in V. By a rule we mean here a subset of $2^S \times S$ usually written in the form:

$$\frac{\alpha_1, \alpha_2, \ldots, \alpha_k}{\beta}$$

The symbol

$$\frac{\alpha, F(\alpha, \beta)}{\beta}$$

will denote the rule $r = \{(\{\alpha, F(\alpha, \beta)\}, \beta) : \alpha \in S, \beta \in S\}$.
The rule r in $\mathcal{S} = (S, F)$ is reliable in the algebra $\mathcal{E} = (E, f)$ if and only if whenever premises of this rule belong to set of tautologies over $\mathcal{E}$, then the conclusion of rule r belongs to the set of tautologies too.
Rule r over the language $\mathcal{S} = (S, F)$ is *normal* in the algebra $\mathcal{E} = (E, f)$ if and only if, for every function $v : V \to E$ and every premise α and conclusion β of the rule r,

if $\|\alpha\|\,(v) \in E^*$ then $\|\beta\|(v) \in E^*$ too.

Of course, if the rule r is normal in $\mathcal{E}$, then it is reliable.

Let F be a binary logical connective. We shall say that F possesses the property $(*)$ if

for every valuation $v : V \to E$ and for all a and b in S, the following two conditions are equivalent:

(1) $\|F(\alpha, \beta)\|(v)$ belongs to E^*;

(2) $\|\alpha\|(v) < p$ or $r \leq \|\beta\|(v)$.

Lemma 1

If F is a binary logical connective possessing the property $(*)$, then the following two conditions are equivalent:

(1) the rule $\frac{\alpha, F(\alpha,\beta)}{\beta}$ is normal

(2) the conditions $x \geq r$ and $y < r$ imply $f(x, y) < r$ – for all x, $y \in E$.

Proof. Suppose that (1) does not hold.

Hence there is a valuation $v : V \to E$ and formulae α, β such that $\|\alpha\|(v) \geq r$, $\|F(\alpha,\beta)\|(v) \geq r$, and $\|\beta\|(v) < r$. We have $\|F(\alpha,\beta)\| = f(x,y)$ for $x = \|\alpha\|(v), y = \|\beta\|(v)$, contradicts (2). It is obvious that (1) implies (2).

2. Some homomorphism between matrices.

Let $\mathfrak{M}_1 = (E_1, E_1^*, f_1)$ and $\mathfrak{M}_2 = (E_2, E_2^*, f_2)$ be logical matrices over the same language $\mathcal{S} = (S, F)$. We say that $\mathfrak{M}_1$ is *homomorphic* to $\mathfrak{M}_2$ if and only if there is *a* homomorphism h from $\mathfrak{M}_1$ to $\mathfrak{M}_2$.

A *homomorphism* from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ is any function $h : E_1 \to E_2$ such that

(h_1) $h(E_1^*) \subseteq E_2^*$

(h_2) the diagram

$$\begin{array}{ccc} E_1^2 & \xrightarrow{\ f_1\ } & E_1 \\ {\scriptstyle h\times h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ E_2^2 & \xrightarrow[\ f_1\ ]{} & E_2 \end{array}$$

is commutative.

Example.
Let the function $f : E^2 \to E$ be such that $f(x,y) \in E^*$ if $x \leq y$ and $f(x,y) \notin E^*$ if $x \leq y$. Then the algebra is homomorphic to the algebra $E_2 = (\{0,1\},\{1\},g)$, where

$$g(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Corollary 1.
If an algebra $\mathcal{E} = (E, E^*, f)$ is homomorphic to the algebra $I = (\{0,1\},\{1\},g)$, then the rule

$$\frac{a, F(a,b)}{b}$$

is normal in the algebra $\mathcal{E}$.

Lemma 2.

Suppose that for $i = 1,2$, $E_i = 1,2,\ldots,n_i$, and $f_i : E_i^2 \to E_i$ is defined by

$$f_i(x,y) = \begin{cases} n_i & \text{if} \quad x \leq y \\ a, a < n_i & \text{if} \quad x > y \end{cases}$$

Then algebras $\mathcal{E}_1$ and $\mathcal{E}_2$ are not homomorphic for any $n_1 \neq n_2$.

Proof.

Suppose that algebras $\mathcal{E}_1$, and $\mathcal{E}_2$ are homomorfic, $n_1 > n_2$ and let the function $h : E_1 \to E_2$ be a homomorphism of algebra $\mathcal{E}_1$ into algebra $\mathcal{E}_2$. Then there exists $h(x) = h(y) = z$. Of course, $f_1(y,z) < n_1$ and $h(f_1(y,x)) < n_2$. However, $h(f_1(y,x)) = f_2(h(y),h(x)) = f_2(z,z) = n_2$, which is impossible. ■

Corollary 2.

If $f(x,y) = n$ for $x \leq y$ and there exists $x \in E$ such that $f(x+1,x) = n$ then the algebra $E = (E,\{n\},f)$ is homomorphic to the algebra $\mathcal{E}' = (E - \{n\},\{n-1\},f_1)$, where $f_1 = f \mid E - \{n\}$.

3. Functionally non–complete logic.

The algebra $\mathcal{A} = (A,\{f_1 : i \leq k\})$, where $k \in N$ and $k > 0$, is *functionally non–complete* if and only if there exists a function
$g : A^m \to A$ which is not defined by the set of functions $\{f_i : i \leq k\}$.

We say that the logic constructed over the language $\mathcal{S} = (S,F)$ is *functionally non–complete* if and only if the language $\mathcal{S} = (S,F)$ has an adequate model in functionally non–complete algebra $\mathcal{E} = (E,f)$.

An example of functionally non–complete three–valued logics is the Łukasiewicz's three–valued implication–negation calculus. The connectives of implication and negation are interpreted as the functions $c(:,:)$ and $n(:)$, respectively, and are defined by the following tables:

c	1	2	3
1	3	3	
2	2	3	3
3	1	2	3

x	$n(x)$
1	3
2	2
3	1

It is well known that the set of the functions $\{c,n\}$ defined above is functionally non – complete, but the set of tautologies over $\mathcal{E} = (\{1,2,3\},\{3\},\{c,n\})$

is finitely axiomatizable by the rule

$$\frac{\alpha, C(\alpha,\beta)}{\beta},$$

where c is the Łukasiewicz's implication.
Of course, the reduct of the logic over the functionally non–complete algebra in which the rule is not reliable is not finitely axiomatizable by this rule.

Can we claim that if the deriving rule

$$\frac{\alpha, F(\alpha,\beta)}{\beta}$$

is normal in the algebra $\mathcal{E}_3 = (\{1,2,3\},\{3\},\{f,n\})$, then the reduct of the logic over $\mathcal{E}$ is finitely axiomatizable by this rule. The answer to this qestion is negative. Let the functions f and n be defined by the following tables

f	1	2	3
1	2	2	2
2	2	2	2
3	2	2	2

x	$n(x)$
1	3
2	1
3	2

The set of tautologies over the $\mathcal{E}_3 = (\{1,2,3\},\{f,n\})$ is a denumerable set of formulae having the form $N^2\alpha, N^5\alpha, \ldots, N^{2n+1}\alpha, \ldots$ where $\alpha = F(\beta,\gamma)$ for all $\beta,\gamma \in S$,

$$N^1(\alpha) = N(\alpha)$$

$$N^{k+1}(\alpha) = N(N(\alpha)),$$

the interpretation of N is the function n, and the interpretation of F is the function f. It is clear that this reduct is not finitely axiomatizable by the deriving rule. Thus we have proved the existence of both the axiomatizable and non–axiomatizable reduct of logic over the algebra $\mathcal{E}_3$ by the deriving rule:

$$\frac{\alpha, C(\alpha,\beta)}{\beta},$$

Everywhere in the above examples, the sets of tautologies are non empty. One can still choose among the reducts of the three–valued logic with a single connective a class of reducts which are homomorphic to reducts of some two–valued logic, so that the axiomatization reduces to an axiomatization of the reduct of the two–valued logic. This axiomatization is presented in the paper [7]. In order to prove the above we will give some following facts. An algebra $\mathcal{E}' = (E', f \mid E')$ is a *proper subalgebra* of $\mathcal{E} = (E, f)$ if and only if $0 \neq E' \subseteq E$ and the set E' is closed with respect to the operation $f \mid E'$, where $f \mid E'$ is the restriction of f to the set E'.

A relation $e \subseteq E \times E$ is a ***proper congruence*** of the algebra $\mathcal{E}$ if and only if r is not the identity relation, and $(x_1, x_2) \in r$, $(y_1, y_2) \in r$ implies $(f(x_1, y_1), f(x_2, y_2)) \in r$ for every $x_1, x_2, y_1, y_2 \in E$.

A function $s : E \xrightarrow{1-1} E$ is a proper automorphism of the algebra $\mathcal{E}$ if and only if s is not the identity function, and for every $x, y \in E$,

$$s(f(x,y)) = f(s(x), s(y))$$

Theorem 1.

Let $L(\mathcal{E})$ be a set of tautologies over $\mathcal{E}$. If $L(\mathcal{E}) \neq \emptyset$ and the algebra $\mathcal{E}$ is functionally non–complete, then at least one of the following conditions is satisfied:

(1) the only proper subalgebras of $\mathcal{E}$ are

$(\{1,3\}, \{3\}, f \mid \{1,3\})$,

$(\{2,3\}, \{3\}, f \mid \{2,3\})$ and

$(\{3\}, \{3\}, f \mid \{3\})$,

(2) the algebra $\mathcal{E}$ has only one automorphism which is defined by the following table:

x	1	2	3
$s(x)$	2	1	3

,

(3) the algebra $\mathcal{E}$ has congruences which are defined by the following decompositions of the set E :

(a) $E\{1,2\} \cup \{3\}$

(b) $E = \{1\} \cup \{2,3\}$

(c) $E = \{1,2,3\} \cup \{2\}$

Proof. Suppose that none of conditions (1), (2), (3) is fulfilled. Then by the Rosenbera – Russeau theorem the algebra $\mathcal{E}$ is functionally complete, which is impossible. ▪

Lemma 3.

If the algebra $\mathcal{E}$ has the congruence defined by decomposition (i) and $L(\mathcal{E}) = 0$, then $x, y \in \{1,2\}$ implies $f(x,y) = 3$

Proof. Trivial. ▪

Lemma 4.

Let the algebra $\mathcal{E}$ satisfies the assumption of lemma 3. If $x \in \{1,2\}$ or $y \in \{1,2\}$, then $f(x,y)$ belongs to exactly one of the sets $\{1,2\}$ or $\{3\}$.

Proof. Trivial. ■

Lemma 5.

Let $\mathcal{E} = (\{1,2,3\},\{3\},f), f : E^2 \to E$. If the algebra $\mathcal{E}$ has the automorphism $(*)$, then the algebra $\mathcal{E}$ has a congruence defined by the decomposition (i).

Proof. Trivial. ■

Lemma 6.

If the algebra $\mathcal{E} = (\{1,2,3\},\{3\},f)$ has a congruence defined by the decomposition (i), and the constant 3 is definable in the algebra $\mathcal{E}$ then the algebra $\mathcal{E}$ is homomorphic with exactly one of the five algebras $(\{0,1\},\{1\},f_i)$ for $i = 1,2,3,4,5$, where the f_i's are defined by following tables:

f_1	0	1
0	1	0
1	0	0

f_2	0	1
0	1	0
1	1	1

f_3	0	1
0	1	1
1	0	1

f_4	0	1
0	1	0
1	0	1

f_5	0	1
0	1	1
1	1	0

Proof. The proof is based on Corollary 2. The homomorphism h is defined in the following way:

$$h(x) = \begin{cases} 0 & \text{if } x \in \{1,2\} \\ 1 & \text{if } x = 3 \end{cases} \quad ■$$

Theorem 2.

Let $E = (\{1,2,3\},\{3\},f)$ include either the automorphism $(*)$ or the congruence defined by the decomposition (i), and let the set $L(\underline{E})$ be non-empty. Then every logic L over the $\mathcal{E}$ has the set of tautologies identical to a certain reduct of the two-valued sentence logic.

Proof. Immediately from the lemma 4,5,6. ■

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Abstract

The present paper contains an investigation of functionally non– complete reducts of three–valued logics. The axiomatization of some three–valued logics may be reduced to the axiomatization of the two–valued logics.

Algebraiczna charakteryzacja pewnych reduktów logik trójwartościowych

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Streszczenie

Niniejsza praca zawiera pewne rozważania funkcyjnie niepełnych reduktach logik trójwartościowych. Aksjomatyzacja pewnych logik trójwartościowych może być zredukowana do aksjomatyzacji określonych logik dwuwartościowych.