Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka IX, Częstochowa 2003

Remarks about the Replacement of School Induction Definitions by Normal Definitions

Grzegorz Bryll, Grażyna Rygał

Pupils and teachers often ask themselves a question: can induction definitions be replaced in an equivalent way by normal definitions? In this paper we present a method of replacement of induction definitions by normal definitions illustrating the given theorems by a few examples. From the viewpoint of the set theory operations and relations can be treated as certain sets. We discuss a method of replacement of an induction definition of the given set by a normal definition of this set. An induction definition of a set A has in general the following form (compare with [2]):

D1. A set A is the least one from among the sets X satisfying the conditions:

$\mathrm{W}_1(X):\ a_1, \ldots, a_n \in X$ (the starting condition),
$\mathrm{W}_2(X):\ x_1, \ldots, x_n \in X \Rightarrow f(x_1, \ldots, x_n) \in X$ (the induction condition).

For example, the following definition of a set of all sensible expressions of implication-negation propositional calculus has such a typical form:

D2. The set S of sensible expressions is the least one from among the sets X satisfying the conditions:

$$\mathrm{W}_1(X):\ p_1, p_2, \ldots \in X,$$
$$\mathrm{W}_2(X):\ \alpha, \beta \in X \Rightarrow \sim\alpha,\ \alpha \to \beta \in X$$

($p_1, p_2, \ldots$ are the sentential variables).

Similarly, the induction definition of a family Fin of all finite subsets of an infinite set V has the following form (compare with [2]):

D3. A family Fin is the least one from among the families $\mathfrak{A}$ ($\mathfrak{A} \subseteq 2^{\mathrm{V}}$) satisfying the conditions:

$$\mathrm{W}_1(\mathfrak{A}):\ \emptyset \in \mathfrak{A},$$
$$\mathrm{W}_2(\mathfrak{A}):\ \underset{X}{\forall}\ \underset{a}{\forall} \left[\mathrm{X} \in \mathfrak{A} \wedge \mathrm{a} \in \mathrm{V} \Rightarrow \mathrm{X} \cup \{\mathrm{a}\} \in \mathfrak{A}\right].$$

The normal definition of the set A discussed in definition 1 has the form:

D4. $x \in A \Leftrightarrow \underset{X}{\forall}\Big[W_1(X) \wedge W_2(X) \Rightarrow x \in X\Big].$

It can be shown that (compare with [2]):

Theorem I. The expression D4 is equivalent to conjunction of three conditions:

(a) $W_1(A)$,
(b) $W_2(A)$,
(c) $\underset{X}{\forall}\Big[W_1(X) \wedge W_2(X) \Rightarrow A \subseteq X\Big]$. (the minimization condition)

To verify that the set A specified by definition D4 is a function the existence and uniqueness conditions should be proved. For three-argument relation A being a function of two variables the existence and uniqueness conditions take the form:

(i) $\underset{a,b}{\forall}\ \underset{c}{\exists}\ (a, b, c) \in A,$ (the existence condition)

(j) $\underset{a,b,c,d}{\forall}\ [(a, b, c), (a, b, d) \in A \Rightarrow c = d].$ (the uniqueness condition)

Example 1. In a set of natural numbers $\mathbb{N}$ the induction definition of the relation less than $<$ ($< \subseteq \mathbb{N} \times \mathbb{N}$) is as follows:

D5a. $0 < x,$
b. $x < y \Rightarrow s(x) < s(y),$

where s is a function of a successor $(s(n) = n + 1)$.
Then, the conditions W_1 and W_2 take the form

$W_1(<)$: $(0, x) \in <,$
$W_2(<)$: $(x, y) \in < \Rightarrow (s(x), s(y)) \in < .$

Using the normal definition the relation ,,<" can be defined in the following way:

D6. $(x, y) \in < \Leftrightarrow \underset{X}{\forall}\Big[W_1(X) \wedge W_2(X) \Rightarrow (x, y) \in X\Big]$

or

D6′. $x < y \Leftrightarrow \underset{X}{\forall}\Big[\underset{x}{\forall}(0, x) \in X \wedge \underset{x,y}{\forall}\big((x, y) \in X \Rightarrow (s(x), s(y)) \in X\big) \Rightarrow$
$\Rightarrow (x, y) \in X\Big].$

It should be noted that in the natural number arithmetic the relation $<$ is generally defined using the normal definition:

D7. $(x, y) \in < \Leftrightarrow \underset{z \neq 0}{\exists}(x + z = y).$

In the work [3] (pp. 26–28) the following theorem is proved using the complete induction:

Theorem II. If $g(n_2, \ldots, n_k)$ and $h(n_1, n_2, \ldots, n_k, n_{k+1})$ are operations in a set of natural numbers, then there exists one and only operation $f(n_1, \ldots, n_k)$ satisfying the conditions:

$$\mathrm{W}_1(f): \quad f(0, n_2, \ldots, n_k) = g(n_2, \ldots, n_k),$$
$$\mathrm{W}_2(f): \quad f(s(n), n_2, \ldots, n_k) = h(n, n_2, \ldots, n_k, f(n, n_2, \ldots, n_k)).$$

Introducing the notation $\mathbf{a} = (n_2, \ldots, n_k)$ these conditions can be written as follows:

$$\mathrm{W}'_1(f): \quad f(0, \mathbf{a}) = g(\mathbf{a}),$$
$$\mathrm{W}'_2(f): \quad f(s(n), \mathbf{a}) = h(n, \mathbf{a}, f(n, \mathbf{a})),$$

where $s(n) = n + 1$.

The normal definition of an operation f satisfying the conditions $\mathrm{W}'_1(f)$, $\mathrm{W}'_2(f)$ has the form:

D8. $$y = f(n, \mathbf{a}) \Leftrightarrow \forall_z \Big[\mathrm{W}'_1 \wedge \mathrm{W}'_2 \Rightarrow y = z(n, \mathbf{a}) \Big].$$

If the k-ary operation is treated as a relation T between $k + 1$ independent variables, then the conditions $\mathrm{W}'_1(f)$, $\mathrm{W}'_2(f)$ can be written in the form

$$\mathrm{W}''_1(f): \quad (0, \mathbf{a}, g(\mathbf{a})) \in T,$$
$$\mathrm{W}''_2(f): \quad \forall_{u,v} \Big[(u, \mathbf{a}, v) \in T \Rightarrow (s(u), \mathbf{a}, h(u, \mathbf{a}, v)) \in T \Big].$$

The normal definition of the relation T (or the operation f) takes the form:

D8′. $$(x, \mathbf{a}, y) \in T \Leftrightarrow \forall_X \Big[\mathrm{W}''_1 \wedge \mathrm{W}''_2 \Rightarrow (x, \mathbf{a}, y) \in X \Big].$$

Example 2. Using induction the operation of addition in a set of natural numbers $\mathbb{N}$ can be defined by a function of a successor s in the following manner:

D9 a. $m + 0 = m, \quad (\mathrm{W}_1(+))$

b. $m + s(n) = s(m + n). \quad (\mathrm{W}_2(+))$

The normal definition of the operation „+" is as follows:

D10. $$y = u + v \Leftrightarrow \forall_z \Big[\mathrm{W}_1(z) \wedge \mathrm{W}_2(z) \Rightarrow y = z(u, v) \Big],$$

where

$$\mathrm{W}_1(z): \quad \forall_m \Big[z(m, 0) = m \Big],$$
$$\mathrm{W}_2(z): \quad \forall_{m,n} \Big[z(m, s(n)) = s(z(m, n)) \Big].$$

If the operation „+" is treated as a relation between three independent variables, then the conditions (a) and (b) of the definition D9 take the form:

$$\mathrm{W}_1'(+): \quad (m, 0, m) \in +,$$
$$\mathrm{W}_2'(+): \quad \underset{u,v}{\forall}\Big[(m, u, v) \in + \Rightarrow (m, s(u), s(v)) \in +\Big].$$

The normal definition of the relation + is as follows:

D10′. $(m, x, y) \in + \;\Leftrightarrow\; \underset{X}{\forall}\Big[\mathrm{W}_1' \wedge \mathrm{W}_2' \Rightarrow (m, x, y) \in X\Big].$

A proof that the relation „+" due to definition D10′ is a function can be found in the work [1, pp. 339-344] (see also [4]).

Example 3. The induction definition of the multiplication operation in a set of natural numbers using the addition operation and the operation of a successor has the form:

D11 a. $m \cdot 0 = m,$

b. $m \cdot s(n) = (m \cdot n) + m.$

The normal definition is as follows:

D12. $y = u \cdot v \Leftrightarrow \underset{z}{\forall}\Big[\mathrm{W}_1(z) \wedge \mathrm{W}_2(z) \Rightarrow y = z(u, v)\Big],$

where

$$\mathrm{W}_1(z): \quad \underset{m}{\forall}\Big[z(m, 0) = 0\Big],$$
$$\mathrm{W}_2(z): \quad \underset{m,n}{\forall}\Big[z(m, s(n)) = z(m, n) + m\Big].$$

Treating the binary operation

„•" as a relation between three independent variables the starting condition and the induction condition can be written as:

$$\mathrm{W}_1'(\bullet): \quad (m, 0, 0) \in \bullet,$$
$$\mathrm{W}_2'(\bullet): \quad \underset{u,v}{\forall}\Big[(m, u, v) \in \bullet \Rightarrow (m, s(u), v + m) \in \bullet\Big].$$

Then the normal definition of the relation „•" takes the form:

D12′. $(m, x, y) \in \bullet \;\Leftrightarrow\; \underset{X}{\forall}\Big[\mathrm{W}_1' \wedge \mathrm{W}_2' \Rightarrow (m, x, y) \in X\Big].$

Example 4. The factorial function has the following induction definition:

$$0! = 1; \qquad s(n)! = n! \cdot s(n), \qquad (n \in \mathbb{N}).$$

The normal definition is as follows:

$$y = n! \Leftrightarrow \underset{z}{\forall}\Big[\mathrm{W}_1(z) \wedge \mathrm{W}_2(z) \Rightarrow y = z(n)\Big],$$

where

$$\mathrm{W}_1(z): \quad z(0) = 1,$$
$$\mathrm{W}_2(z): \quad \underset{n\in\mathbb{N}}{\forall} \Big[z(s(n)) = z(n) \cdot s(n) \Big].$$

Example 5. Exponentiation with natural power is defined by induction:

$$a^0 = 1; \qquad a^{s(n)} = a^n \cdot a, \qquad (n \in \mathbb{N},\ a > 0,\ a \in R).$$

The normal definition has the form:

$$y = a^n \iff \underset{z}{\forall} \Big[\mathrm{W}_1(z) \wedge \mathrm{W}_2(z) \Rightarrow y = z(n) \Big],$$

where

$$\mathrm{W}_1(z): \quad z(0) = 1,$$
$$\mathrm{W}_2(z): \quad \underset{n\in\mathbb{N}}{\forall} \Big[z(s(n)) = z(n) \cdot a \Big].$$

Example 6. The predecessor operation P in a set of natural numbers has the following induction definition:

$$\begin{cases} P(0) = 0, \\ P(s(n)) = n, \qquad n \in \mathbb{N}. \end{cases}$$

The normal definition is:

$$y = P(n) \iff \underset{z}{\forall} \Big[\mathrm{W}_1(z) \wedge \mathrm{W}_2(z) \Rightarrow y = z(n) \Big],$$

where

$$\mathrm{W}_1(z): \quad z(0) = 0,$$
$$\mathrm{W}_2(z): \quad \underset{n\in\mathbb{N}}{\forall} \Big[z(s(n)) = n \Big].$$

If the predecessor operation is defined by the definition

$$P(n) = \begin{cases} 0, & \text{if } n = 0, \\ n - 1, & \text{if } n > 0, \end{cases}$$

then the normal definition of this operation takes the form:

$$y = P(n) \iff \Big[(n = 0 \wedge y = 0) \vee (n > 0 \wedge y = n - 1) \Big].$$

Example 7. The operation of bounded subtraction „$\dot{-}$" in a set $\mathbb{N}$ can be defined by induction due to the following formulae:

$$\begin{cases} m \dot{-} 0 = m, \\ m \dot{-} s(n) = P(m \dot{-} n), \qquad n \in \mathbb{N}, \end{cases}$$

where P is the predecessor operation.

The normal definition has the form:

$$y = u \dot{-} v \;\Leftrightarrow\; \forall_{z} \Big[\mathrm{W}_1(z) \wedge \mathrm{W}_2(z) \Rightarrow y = z(u, v) \Big],$$

where

$$\mathrm{W}_1(z): \quad \forall_{m \in \mathbb{N}} \Big[z(m, 0) = m \Big],$$

$$\mathrm{W}_2(z): \quad \forall_{m,n} \Big[z(m, s(n)) = P(z(m, n)) \Big].$$

The operation „$\dot{-}$" can also be defined without induction:

$$x \dot{-} y = \begin{cases} 0, & \text{if } x \leq y, \\ x - y, & \text{if } x > y, \end{cases}$$

with the operation „$-$" defined as follows:

$$x > y \Rightarrow \Big[u = x - y \;\Leftrightarrow\; u + y = x \Big].$$

Then the normal definition of the operation „$\dot{-}$" takes the form:

$$u = x \dot{-} y \;\Leftrightarrow\; \Big[(x \leq y \wedge u = 0) \vee (x > y \wedge u = x - y) \Big].$$

Example 8. The induction definitions of arithmetic and geometric sequences have the form:
An arithmetic sequence with the first term a and a common difference r:

$$\begin{cases} f(0) = a, \\ f(s(n)) = f(n) + r. \end{cases}$$

A geometric sequence with the first term a and a common ratio q:

$$\begin{cases} g(0) = a, \\ g(s(n)) = g(n) \cdot q. \end{cases}$$

The normal definitions of these sequences are the following:

$$y = f(n) \;\Leftrightarrow\; \forall_{z} \Big[z(0) = a \wedge \forall_{n \in \mathbb{N}} \Big(z(s(n)) = z(n) + r \Big) \Rightarrow y = z(n) \Big],$$

$$y = g(n) \;\Leftrightarrow\; \forall_{z} \Big[z(0) = a \wedge \forall_{n \in \mathbb{N}} \Big(z(s(n)) = z(n) \cdot q \Big) \Rightarrow y = z(n) \Big].$$

References

[1] T. Batóg, *Podstawy logiki.* Wyd. 3, Wydawnictwo Naukowe UAM, Poznań, 1999.

[2] A. Grzegorczyk, *Zarys logiki matematycznej.* Biblioteka Matematyczna t. 20, PWN, Warszawa, 1961.

[3] W. Sierpiński, *Arytmetyka teoretyczna.* Biblioteka Matematyczna t. 7, PWN, Warszawa, 1955.

[4] J. Słupecki, K. Hałkowska, K. Piróg–Rzepecka, *Elementy arytmetyki teoretycznej.* Biblioteczka Matematyczna t. 38, WSiP, Warszawa, 1980.

Institute of Mathematics and Computer Science
Opole University
Poland

Institute of Mathematics and Computer Science
Pedagogical University of Częstochowa
Poland