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On Usability of Ordering and Equivalence Relations to Define and to Ground Some Mathematical Notions in Secondary Schools

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Many important mathematical notions can be introduced by means of the notions of supremum, infimum and equivalence class. Our considerations refer to mainly pupils of mathematical sections.

1 Supremum and infimum of ordered sets

A relation "$\leq$" is said to be an order in a set X if the following hold[1]:

(a) $\forall_{x \in X}(x \leq x)$ (reflexive),

(b) $\forall_{x,y \in X}(x \leq y \wedge y \leq x \Rightarrow x = y)$ (antisymmetry),

(c) $\forall_{x,y,z \in X}(x \leq y \wedge y \leq z \Rightarrow x \leq z)$ (transitive).

A set of majorants (upper bounds) $M(A)$ and a set of minorants (lower bounds) $m(A)$ of any subset A of an ordered set $(X, \leq)$ may be defined as follows:

$M(A) = \{y \in X : \forall_{x \in A}(x \leq y)\}$,
$m(A) = \{y \in X : \forall_{x \in A}(y \leq x)\}$.

For some A sets $M(A)$ and $m(A)$ may be empty.

The notions of the minimal element and the maximal element of a set B ($B \subseteq X$) are defined in the following way:

(a) x_0 is the least element in a set $B \Leftrightarrow x_0 \in B \wedge \forall_{x \in B}(x_0 \leq x)$,

(b) x_0 is the greatest element in a set $B \Leftrightarrow x_0 \in B \wedge \forall_{x \in B}(x \leq x_0)$.

Now we introduce the notions of the least upper bound (supremum) and the greatest lower bound (infimum) of any set A ($A \subseteq X$)[2]:

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[1] Sometimes a relation "$\leq$" satisfying the conditions (a) – (c) is called a partial order.

[2] There exist three different and nonequivalent definitions of these notions. We adopt the definitions which concern any ordered set (see [2]).

(a) the least upper bound of a set A is the least element of the set $M(A)$ (when it exists),

(b) the greatest lower bound of a set A is the greatest element of the set $m(A)$ (when it exists).

These definitions can be rewritten in the following forms:

(a) $x_0 = \sup A \Leftrightarrow \forall_{x\in A}(x \leq x_0) \wedge \forall_{y\in X}[\forall_{x\in A}(x \leq y) \Rightarrow x_0 \leq y]$,

(b) $x_0 = \inf A \Leftrightarrow \forall_{x\in A}(x_0 \leq x) \wedge \forall_{y\in X}[\forall_{x\in A}(y \leq x) \Rightarrow y \leq x_0]$.

Examples. One can consider the following definitions in a metric space (X, ϱ) with a metric ϱ:

1. The diameter diam(A) of a set A ($A \subseteq X$):

$$\text{diam}e(A) = \begin{cases} \sup_{x,y\in A} \varrho(x,y) & \text{when } A \neq \emptyset, \\ 0 & \text{when } A = \emptyset. \end{cases}$$

2. The distance $d(x, A)$ between a point $x \in X$ and a nonempty set A ($A \subseteq X$):

$$d(x, A) = \inf_{y\in A} \varrho(x, y).$$

3. The distance $s(A, B)$ between any two nonempty subsets A, B of X:

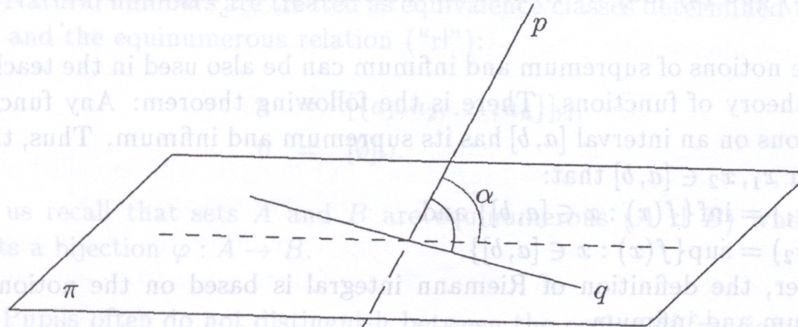
$$s(A, B) = \inf_{x\in A, y\in B} \varrho(x, y).$$

It is easy to notice that the above definitions are related to subsets of nonnegative real numbers with an ordinary order. Thus, the notions: the distance between a point and a line, the distance between a point and a plane, the diameter of a circle (of a figure), the distance between two skew lines or the distance between two circles, may be defined by the notions of an ordinary metric and the supremum (infimum). For instance, two circles are tangent if the distance between them is zero and they are different. The supremum (when it exists) of lengths of all broken lines inscribed into a curve may be treated as the length of that curve.

The measure of an inclination angle of a line p to a plane π ($p \not\parallel \pi$) is defined by the infimum as follows:

$$\alpha = \inf_{q\subset\pi}(\angle p, q),$$

where the point of intersection of p and π belongs to every line q.



A set of convex and limited figures on a plane can be used to define an area of a figure. For example, the area of a circle is the supremum of the set of all areas of convex polygons included in this circle.

The notions of the greatest common divisor (GCD) and the smallest common multiple (SCM) can be defined in the set of all natural numbers in the following way:

$GCD(a,b) = \sup d(a,b)$,

$SCM(a,b) = \inf m(a,b)$,

where the symbol $d(a,b)$ and $m(a,b)$ denote the set of all common divisors of a and b and the set of all common multiples of a and b respectively (similarly, the definition works for three and more natural numbers).

For any two subsets A, B of X ordered by the inclusion relation one can define the operation of union "∪" and intersection "∩" as follows:

$$A \cup B = \sup\{A, B\}, \quad (1)$$

$$A \cap B = \inf\{A, B\}. \quad (2)$$

Analogically, for a family of sets $\{A_1, A_2, \ldots, A_n\}$ we have:

$$A_1 \cup A_2 \cup \ldots \cup A_n = \sup\{A_1, A_2, \ldots, A_n\},$$

$$A_1 \cap A_2 \cap \ldots \cap A_n = \inf\{A_1, A_2, \ldots, A_n\}.$$

The definitions (1) and (2) take other forms in secondary schools. In many mathematical textbooks one can find the following ones:

$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}.$$

Then (1) and (2) may be treated as theorems of the algebra of sets.

The notions of supremum and infimum can be also used in the teaching in the theory of functions. There is the following theorem: Any function continuous on an interval $[a, b]$ has its supremum and infimum. Thus, there are such $x_1, x_2 \in [a, b]$ that:
$f(x_1) = \inf\{f(x) : x \in [a, b]\}$ and
$f(x_2) = \sup\{f(x) : x \in [a, b]\}$.
Moreover, the definition of Riemann integral is based on the notions of supremum and infimum.

2 Equivalence classes

A relation "$\equiv$" is called an equivalence relation in a set A if the following hold:

(a) $\forall_{x \in A}(x \equiv x)$ (reflexive),

(b) $\forall_{x,y \in A}(x \equiv y \Rightarrow y \equiv x)$ (symmetry),

(c) $\forall_{x,y,z \in A}(x \equiv y \wedge y \equiv z \Rightarrow x \equiv z)$ (transitive).

An equivalence class $[a]_{\equiv}$ determined by an element $a \in A$ and a relation "$\equiv$" can be defined in the following way:
$[a]_{\equiv} = \{b \in A : b \equiv a\}$
The class $[a]_{\equiv}$ consists of the elements that are in the relation "$\equiv$" with the element a. It is known that the set of all equivalence classes is the partition of the set A. It means that the following conditions are true:

(a) $\forall_{a \in A}([a]_{\equiv} \neq \emptyset)$,

(b) $\forall_{a,b \in A}([a]_{\equiv} \neq [b]_{\equiv} \Rightarrow [a]_{\equiv} \cap [b]_{\equiv} = \emptyset)$,

(c) $\bigcup_{a \in A}[a]_{\equiv} = A$.

In secondary schools many mathematical notions can be defined by an equivalence relation and an equivalence class.

For example, the notion of a direction K_l on a plane determined by the line l may be defined in the following way:

$$K_l = \{m : m \parallel l\} = [l]_{\parallel},$$

where the symbol "$\parallel$" denotes the parallel relation.

Natural numbers are treated as equivalence classes determined by finite sets and the equinumerous relation ("rl"):

$$\begin{aligned} n &= [\{a_1, a_2, \ldots, a_n\}]_{rl}, \\ 0 &= [\emptyset]_{rl}. \end{aligned}$$

Let us recall that sets A and B are equinumerous (A rl B) when there exists a bijection $\varphi : A \to B$.

Pupils often do not distinguish between the notion of fraction and the notion of rational number. If we introduce the equivalence relation "$\sim$" (of fraction equality) as follows:

$$\tfrac{a}{b} \sim \tfrac{c}{d} \Leftrightarrow a \cdot d = b \cdot c$$

then one can define any rational number $\frac{a}{b}$ determined by the simplified fraction $\frac{a}{b}$ in the following way:

$$\frac{\boldsymbol{a}}{\boldsymbol{b}} = \left[\frac{a}{b}\right]_{\sim} = \left\{\frac{c}{d} : \frac{c}{d} \sim \frac{a}{b}\right\}$$

Thus, a rational number $\frac{a}{b}$ is the set of all equivalence fractions to $\frac{a}{b}$ in the sense of the relation "$\sim$". Moreover, this number may be represented by an arbitrary element of the set $[\frac{a}{b}]_{\sim}$. We have:

$$\frac{1}{2} = \left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots\right\},$$

$$\frac{2}{3} = \left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\right\}.$$

The notions of "even number" and "odd number" are also connected with the notion of an equivalence class. We define the relation "$\approx_{(2)}$" as follows:

$$a \approx_{(2)} b \Leftrightarrow 2|a - b,$$

where $a, b \in Z$. It is easy to see that this relation is an equivalence relation. The set A of even numbers and the set B of odd numbers have the form:

$$A = [0]_{\approx_{(2)}} \text{ and } B = [1]_{\approx_{(2)}}.$$

Hence, $A = \{x \in Z : x \approx_{(2)} 0\}$ and $B = \{x \in Z : x \approx_{(2)} 1\}$.

Let us notice that the sets of integer numbers, rational numbers and real numbers are constructed by equivalence classes on the university courses in mathematics.

An equivalence class which is determined by a localized vector $\overrightarrow{A_0B_0}$ and the equality relation "=" of localized vectors, i.e.

$$\overrightarrow{AB} = [A_0B_0]_= = \{\overrightarrow{PQ} : \overrightarrow{PQ} = \overrightarrow{A_0B_0}\}$$

where $\overrightarrow{PQ} = \overrightarrow{A_0B_0} \Leftrightarrow |\overrightarrow{PQ}| = |\overrightarrow{A_0B_0}|$ and $\overrightarrow{PQ} \upuparrows \overrightarrow{A_0B_0}$ will be called a free vector.

One can consider relations of the congruence of figures ("≡") and the similarity of figures ("∼") on a plane. Each of them refers to an equivalence class. For example:

$[\triangle ABC]_\equiv = \{\triangle PQR : \triangle PQR \equiv \triangle ABC\}$,

$[\triangle ABC]_\sim = \{\triangle PQR : \triangle PQR \sim \triangle ABC\}$.

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