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# Nonelementary Notes on Elementary Events*

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**Abstract.** Our goal is to present simple examples illustrating the nature and role of elementary events and random variables in probability theory, both classical and operational (fuzzy).

As stated in Płocki [10], in teaching probability we should concentrate on the construction of probability spaces and their properties, and not on the calculation of probability of various strange events (like hitting a bear if we can shoot three times, etc.). On a rather advanced level, Łoś [8] analyzed the constructions of probability spaces in the classical probability. J. Łoś explained the nature and underscored the role of elementary events. Roughly, the events form a Boolean algebra, but some probability properties of the algebra depend on its representation via subsets and this is done via the choice of some fundamental subset of events and the choice of elementary events. *Remember, choice!*

There are situations in which the classical probability model is not quite suitable (quantum physics, fuzzy models, c.f. Dvurečenskij and Pulmanová [3], Frič [5]), and I would like to present simple examples and simple models of such situations. In order to understand the generalizations, let me start with a well-known example of throwing two dice.

**Example 1.** *When throwing two dice, we are in fact interested in events like: the resulting sum $s$ is equal to $k$, the sum $s$ is greater than $k$, or $l \leq s < k$, where $l, k = 2, 3, \ldots, 12$. To describe what is going on, we start with elementary events (called outcomes) and we try to assign them probabilities in a reasonable way. Instead of working with seemingly indecomposable "numerical" outcomes of the form "$s = k$" we (clever mathematicians) define the elementary events as ordered pairs $(i, j)$ - we declare which die of the two dice will " act" independently on the first coordinate stage and then the other die will "act" independently on the second coordinate stage. Further, each pair "has the same probability" hence $p((i, j)) = \frac{1}{36}$ and, consequently, the probability of "$s = k$" is equal to $\frac{n}{36}$, where $n$ is the number*

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*of pairs $(i, j)$ such that $i + j = k$. Finally, the probability of "$l \leq s < k$" is equal to $\frac{n}{36}$, where $n$ is the number of all pairs $(i, j)$ such that $l \leq i + j < k$.*

Now, the sum sending $(i, j) \mapsto i + j = s$ yields a map $s$ of the set $\Omega = \{(i, j); i, j \in \{1, 2, ..., 6\}\}$ of elementary events to the real line $R$ (onto $\{2, 3, ..., 12\} \subset R$); in general such a map is called a **random variable**. More explicitly:

(a) We have two dice: $A$ and $B$.

(b) We carry out two experiments: we throw die $A$ and, independently, we throw die $B$.

(c) We measure (describe) the results: there are 36 relevant mutually exclusive results - ordered pairs of squares with dots, e.g. ($\boxdot$, $\boxdot$), indicating the number of dots on the top side of the die $A$ and the die $B$ (we are not interested in other details - e.g. the location of dice).

(d) We assign a nonnegative number to each pair so that their total sum is 1. Since we assume that each die is regular and the experiments are independent, all numbers are the same: $\frac{1}{36}$ (if a die is irregular, then some other assignment is appropriate). The assignment yields a probability measure on the set of all events (all subsets of pairs).

(e) The sum $s$ sends each pair of dotted squares into the real line $R$ and the preimage $s^{\leftarrow}$ sends " numerical" events (i.e. Borel measurable sets of real numbers) into the original events (subsets of ordered pairs).

(f) The sum sends each probability measure $p$ on the set of all events to the probability measure $p_s$ on the " numerical" events $B \subseteq R$ via $p_s(B) = p(s^{\leftarrow}(B)) = p(\{(i, j); i + j \in B\})$; $p_s$ is called the **distribution** of $s$.

In the classical probability theory, we have a **sample probability space** $(\Omega, \mathrm{S}, p)$, a **random variable** $f : \Omega \longrightarrow R$ (a measure preserving (Borel) measurable function on $\Omega$), and the **"numerical" probability space** $(R, \mathcal{B}(R), p_f)$ such that $f^{\leftarrow} : \mathcal{B}(R) \longrightarrow \mathbf{S}$ is a Boolean homomorhism (preserves set operations: union, intersection, complementation) and $p_f(B) = p(f^{\leftarrow}(B))$. Observe, that $f$ (via he preimage $f^{\leftarrow}$) transforms each probability measure $q$ on $\mathbf{S}$ into a probability measure $q_f$, the **distribution** of $f$ (w.r.t. q). *Remember, $f$ sends elementary events to elementary events (real numbers) and probability measures to probability measures.*

Next we are going to present a situation in which the classical probability theory is not suitable.

**Example 2.** *Consider a digital display - for the simplicity showing only one digit. There are ten objects $O_i$, drawn at random with probability $P_i$, $i = 1, 2, ..., 10$. On our display we can see seven segments $s_i, i = 1, 2, ..., 7$, four vertical (upper left, upper right, lower left, lower right) and three horizontal (upper, middle, lower) and each of them is either "on" or "off". In the classical case, if $O_i$ is drawn, then the display shows its "code", e.g. $O_1$ is indicated by the two right vertical segments "on" and all other segments "off" (the digital version of number one). If there is a "noise", causing that with some probability a segment is "off" instead of being "on", then the display shows not the "code" of the object, but its distorted version, e.g. in a case of $O_1$ it can happen that, instead of two right vertical segments, only one of them is "on" (or none). Then, in fact, there are $2^7$ possible outcomes $\omega_i$, $i = 1, 2, ..., 128$, on the display, each indicating the particular object $O_j$ only with some probability $p_{ij}$, $j = 1, 2, ..., 10$. This sends the outcome $\omega_i$ not to a real number $j \in R$ indicating the occurrence of $O_j$, but to some probability measure $p_i$ on subsets of $\{1, 2, ..., 10\}$ such that $p_i(\{j\}) = p_{ij}$, $p_i(\{j, k\}) = p_{ij} + p_{ik}$, etc.*

From the formal point of view, the situation can be handled by a generalized random variable $f$ sending $\omega_i$, $i = 1, 2, ..., 128$, to a probability measure $p_i$ on the (Borel) measurable subsets of the real line $R$, provided that there is a satisfactory generalized probability theory guaranteeing that $f$ sends each measure $p$ on subsets of $\Omega = \{\omega_1, \omega_2, ..., \omega_{128}\}$ into a probability measure $p_f$ on the (Borel) measurable subsets of $R$ in a "consistent" way. A suitable theory has been (partly) developed and it is known as **operational**, or fuzzy, probability theory; observe that then $f$ has a quantum nature. More information can be found in Bugajski [1, 2] and Gudder [6]. Some related results will be presented at the FSTA 2004 conference by myself and my PhD student M. Papčo (cf. Papčo [9]).

We can identify each elementary event (point) and the degenerated (point) measure concentrated at it. Then, roughly, each **operational random variable** sends probability measures on the sample space to probability measures on the "numerical" probability space and it can happen that (unlike in the classical case) a degenerated measure is sent to a nondegenerated one. The price we have to pay for the generalization is that instead of events represented by sets we have to work with events represented by fuzzy sets. We close with a simple example of an operational random variable.

**Example 3.** *Assume that we randomly choose one of two objects $O_1, O_2$. Assume that on a display there are two independent segments $s_1$ and $s_2$. If $O_1$ is chosen, then we try to activate $s_1$ and if $O_2$ is chosen, then we try to activate (independently) both $s_1$ and $s_2$. However, each time we activate* *a segment, for some reason the segment can remain "off". This yields four possible results on the display:* $(i,j)$, *i,j = 0,1 (e.g.* $(0,1)$ *means that that only the second segment is "on"). Question: if* $(i,j)$ *appears on the display, what can be said about the occurrence of* $O_1, O_2$*? According to the Bugajski-Gudder model, we try to assign each* $(i,j)$ *a reasonable probability that* $O_1$ *resp.* $O_2$ *has been chosen.*

First, we construct a classical probability space $(\Lambda, \mathbf{S}, P)$ which is a model of our experiments. Put $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_6\}$, where $H_1 = \{\lambda_1, \lambda_2\}$ is the event "$O_1$ has been chosen" and $H_2 = \{\lambda_3, \lambda_4, \lambda_5, \lambda_6\}$ is the complementary event "$O_2$ has been chosen". Further, $\{\lambda_1\}$ means that $O_1$ has been chosen and $s_1$ is "on", $\{\lambda_2\}$ means $O_1$ and $s_1$ is "off", $\{\lambda_3\}$ means $O_2$ and both $s_1$ and $s_2$ are "on", $\{\lambda_4\}$ means $O_2$ and $s_1$ is "off" and $s_2$ is "on", $\{\lambda_5\}$ means $O_2$ and $s_1$ is "on" and $s_2$ is "off", $\{\lambda_6\}$ means $O_2$ and both $s_1$ and $s_2$ are "off". Then $S_1 = \{\lambda_1, \lambda_3, \lambda_5\}$ means that $s_1$ is "on", $S_2 = \{\lambda_2, \lambda_4, \lambda_6\}$ means that $s_1$ is "off" $S_3 = \{\lambda_3, \lambda_4\}$ means that $s_2$ is "on", $S_4 = \{\lambda_1, \lambda_2, \lambda_5, \lambda_6\}$ means that $s_2$ is "off".

Second, let $p, q, r$ be positive real numbers less than 1. Let $P(H_1) = p$ and $P(H_2) = 1 - p$, let $q$ be the probability that $s_1$ is "on" whenever we try to activate it, let $r$ be the probability that $s_2$ is "on" whenever we try to activate it, and assume that $s_1$ and $s_2$ are independent. Then $P(\{\lambda_1\}) = pq$, $P(\{\lambda_2\}) = p(1-q)$, $P(\{\lambda_3\}) = (1-p)qr$, $P(\{\lambda_4\}) = (1-p)(1-q)r$, $P(\{\lambda_5\}) = (1-p)q(1-r)$, $P(\{\lambda_6\}) = (1-p)(1-q)(1-r)$ yields a suitable probability measure on the events of our model.

Clearly, we can identify $\omega_1 = (0,0)$ and $\{\lambda_2, \lambda_6\} = S_2 \bigcap S_4$, $\omega_2 = (1,0)$ and $\{\lambda_1, \lambda_5\} = S_1 \bigcap S_4$, $\omega_3 = (0,1)$ and $\{\lambda_4\} = S_2 \bigcap S_3$, $\omega_4 = (1,1)$ and $\{\lambda_3\} = S_1 \bigcap S_3$. Finally, define the (generalized) operational random variable $f$ as a mapping of $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ into the set of all probability measures on the "numerical" field of events $\{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ as follows: $f(\omega_i) = p_i$, where $p_i(\{1\}) = P(H_1|\omega_i)$, $p_i(\{2\}) = P(H_2|\omega_i)$, and $P(H_j|\omega_i)$ is the conditional probability of $O_j$ given $\omega_i$; it can be calculated using the Bayes rule for the probability of causes in $(\Lambda, \mathbf{S}, \mathbf{P})$. Observe that $f(\omega_3)$ and $f(\omega_4)$ are degenerated measures, but $f(\omega_1)$ and $f(\omega_2)$ are nondegenerated probability measures.

**Remark 4.** *Observe that D-posets (Kôpka and Chovanec [7]) and effect algebras (Foulis and Bennett [4]), two (equivalent) types of quantum events defined only recently (cf. Dvurečenskij and Pulmannová [3]), are in their nature similar to the calculation on fingers:* $a \ominus b$ *is defined (it is equal to* $a - b$*) iff* $b \leq a$*, resp.* $a \oplus b$ *is defined (it is equal to* $a + b$*) iff* $a + b \leq 10$.

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