

Routes on a Cubic Grid and on a Chessboard

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Here we present two different problems for investigation: counting the routes on a cubic grid and counting the paths on a chessboard. In both investigations we can use two different strategies: a) the strategy “of jumping” and b) the strategy “of going step-by-step”¹. The first strategy leads to combinatorial formulas and the second to number triangles. We feel the strategy of „going step-by-step” is more important at this moment.

Investigation of routes on a cubic grid

P_1 : Consider a three-dimensional cubic grid (See Fig.1). Here we can move in three directions: to the right, up and to the back. How many different routes are there from the P to a point K?

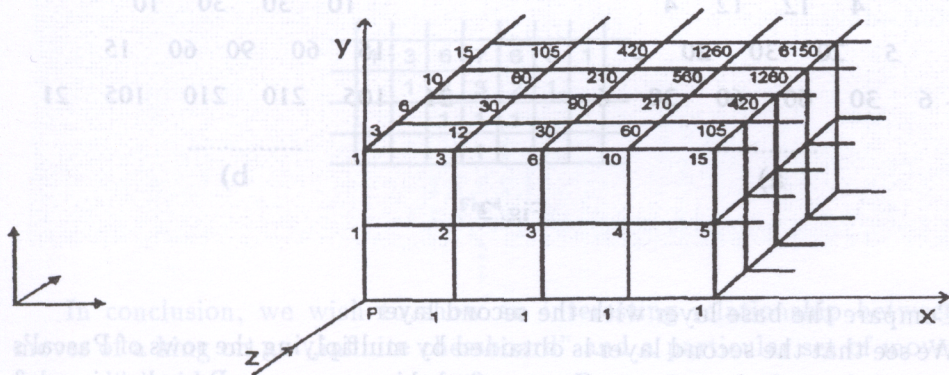


Fig. 1

Solution of P_1

a) The strategy of jumping

We introduce a coordinate system on the grid. Every intersection will have a set of coordinates of the form (x, y, z) , where x, y, z are non-negative

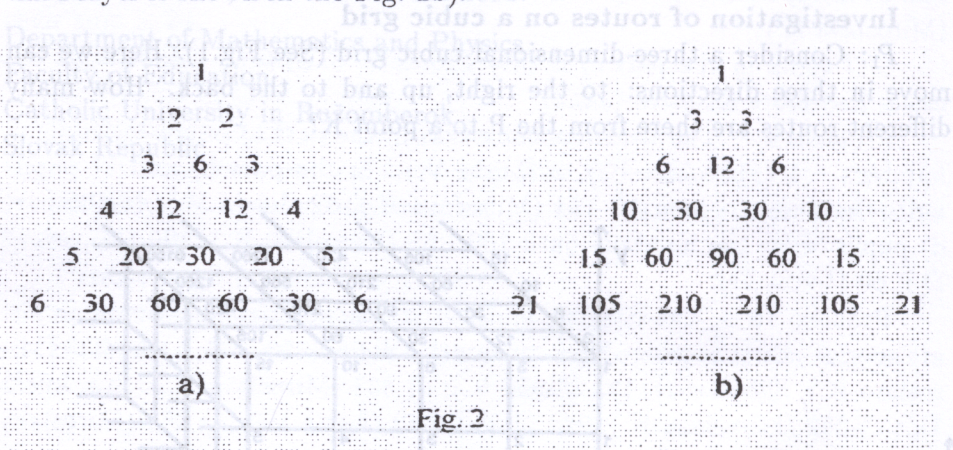
¹The strategies of jumping and going step-by-step are specific to this investigation. See solutions of P_1 .

integers (note the non-traditional orientation of the z -axis in Fig. 1). We will call a move from any point to a neighboring point a „step”. We begin at the origin $P(0,0,0)$. Any route from $P(0,0,0)$ to $K(i, j, k)$ consists of $i + j + k$ steps, of which i have to be parallel to the x -axis, j to the y -axis and k to the z -axis. Thus the number of different routes leading from P to $K(i, j, k)$ is the number of permutations of $i + j + k$ objects, of which i are of one kind, j of a second kind and k of a third kind. This number is equal to

$$\frac{(i + j + k)!}{i!j!k!}$$

b) The strategy of going step-by-step:

Now we are going step by step, first on the „floor” (the points with coordinates of the form $(x, 0, z)$), then one level up (the points with coordinates of the form $(x, 1, z)$) and so forth (see Fig. 1). The number triangle corresponding to the bottom layer of the grid is Pascal’s triangle. The number triangle for the second layer is shown in Fig. 2a) and the triangle for the third layer is shown in the Fig. 2b).



Compare the base layer with the second layer.

We see that the second layer is obtained by multiplying the rows of Pascal’s triangle by 1, 2, 3, 4, 5, ... Can you find this sequence in Pascal’s triangle? Compare the base layer with the third layer.

We see that the third layer is obtained by multiplying the rows of Pascal’s triangle by 1, 3, 6, 10, 15, ... Can you find this sequence in Pascal’s triangle?

Generalize it. That is, answer the question „what is the number triangle for the n -th layer?”

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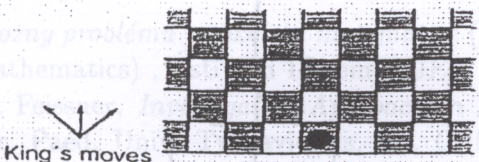


Fig. 3

Investigation of paths on a chessboard

P_2 : The chessboard in Fig. 3 is bounded on the bottom only. A king is placed on a square in the bottom and can only move in three directions - up, up-left and up-right. For each square on Fig. 3 find the number of paths the king may take to this square from the start square, and write this number in the square.

Solution of P_2 :

The strategy of going step by step. The strategy of going step-by-step leads to number triangle in Fig. 4. Investigate this triangle. Can you discover there, for example, any Tribonacci sequence²?

	1	3	6	7	6	3	1
		1	2	3	2	1	
			1	1	1		
				1			

Fig.4

In conclusion, we wish to show an interesting relationship between moves of a king on an „infinite chessboard” and a particular set of moves on a cubic grid.

²The first three terms of the sequence can be chosen arbitrarily, and each succeeding term is the sum of three terms preceding it. The Tribonacci sequence we examine here is: 1, 1, 2, 4, 7, 13, 24, ...

integers (note the non-traditional orientation of the z -axis in Fig. 1). We will call a move from any point to a neighboring point a „step”. We begin at the origin $P(0,0,0)$. Any route from $P(0,0,0)$ to $K(i, j, k)$ consists of $i + j + k$ steps, of which i are to the x -axis, j to the y -axis and k to the z -axis. The number of different routes leading from P to $K(i, j, k)$ is the number of permutations of $i + j + k$ objects, of which i are of one kind, j of a second kind, and k of a third kind. This number is equal to

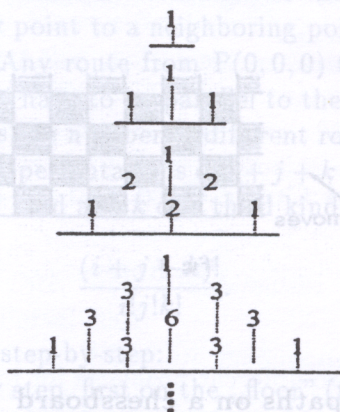
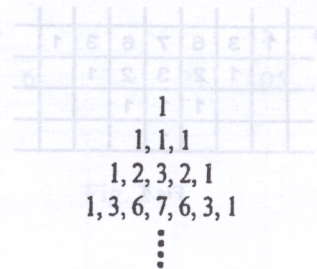


Fig. 5

Cut the cubic grid by planes

Cut the cubic grid (see Fig. 1) by planes π_n through the points $A_n(n,0,0)$, $B_n(0,n,0)$, $C_n(0,0,n)$ for $n = 0, 1, 2, 3, \dots$. The labels lying on π_n create a number triangle, forming the n -th layer of the pyramid (see Fig. 5).

Now compare the number triangles in Fig. 5 with number triangle in Fig. 4. If you add the numbers in each triangle connected by the vertical lines and then use the sums as rows in a new number triangle, you get the following:



which is just the triangle in Figure 4!

A particularly challenging investigation would be to explain what was shown here.

References

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In numerous situations we face the necessity of drawing one element from a finite set. The requirement is that in such an experiment the chances of being selected are equal for each element. A draw satisfying this condition will be called a fair draw. This type of situation occurs, for example, in the case of drawing a half of a playing field before a match or when a question in a competition is selected. It is put to practice with the aid of coins, matches, counting-out rhymes, „wheels of fortune”, receptacles containing pieces of paper with names of people written on them etc.

In this context the following questions arise:

- How should a fair draw of one element from a given n -element set be organised?
- How should a fair draw of two or more elements from a given n -element set be organised?
- How could the fairness of such a draw be verified within the framework of mathematics?

Let us consider the following procedure of drawing.

1. From a group of s people one has to be drawn in such a way that the chances of being selected are equal for each person. Quite often matches are used for this purpose. An organiser of the drawing holds matches in his or her hand; the head of one of the matches has been previously broken off. The matches are held in such a way that the participants of the experiment cannot see these parts of the matches where the head is or has been. Everyone from the group out of which one person has to be selected