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Routes on a Cubic Grid and on a Chessboard

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Here we present two different problems for investigation: counting the routes on a cubic grid and counting the paths on a chessboard. In both investigations we can use two different strategies: a) the strategy "of jumping" and b) the strategy "of going step-by-step"[1]. The first strategy leads to combinatorial formulas and the second to number triangles. We feel the strategy of „going step-by-step" is more important at this moment.

Investigation of routes on a cubic grid

P_1: *Consider a three-dimensional cubic grid (See Fig.1). Here we can move in three directions: to the right, up and to the back. How many different routes are there from the P to a point K?*

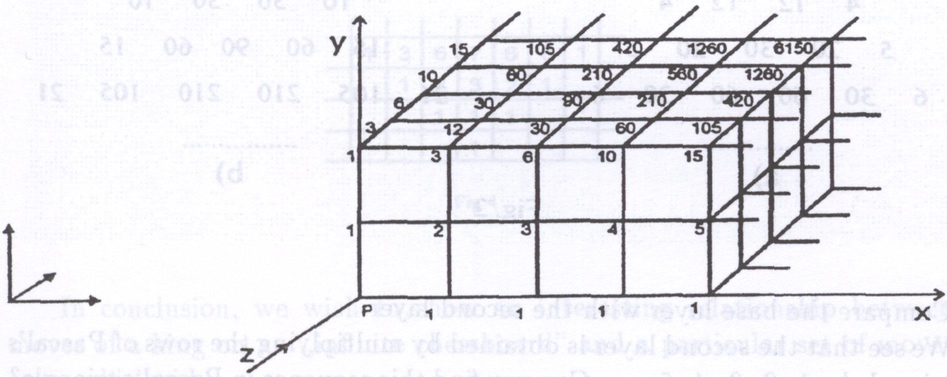

Fig. 1

Solution of P_1

a) The strategy of jumping

We introduce a coordinate system on the grid. Every intersection will have a set of coordinates of the form (x, y, z), where x, y, z are non-negative

[1]The strategies of jumping and going step-by-step are specific to this investigation. See solutions of P1.

integers (note the non-traditional orientation of the z-axis in Fig. 1). We will call a move from any point to a neighboring point a „step" We begin at the origin P(0, 0, 0). Any route from P(0, 0, 0) to K(i, j, k) consists of $i + j + k$ steps, of which i have to be parallel to the x-axis, j to the y-axis and k to the z-axis. Thus the number of different routes leading from P to K(i, j, k) is the number of permutations of $i + j + k$ objects, of which i are of one kind, j of a second kind and k of a third kind. This number is equal to

$$\frac{(i+j+k)!}{i!j!k!}.$$

b) The strategy of going step-by-step:
Now we are going step by step, first on the „floor" (the points with coordinates of the form $(x, 0, z)$), then one level up (the points with coordinates of the form $(x, 1, z)$) and so forth (see Fig. 1). The number triangle corresponding to the bottom layer of the grid is Pascal's triangle. The number triangle for the second layer is shown in Fig. 2a) and the triangle for the third layer is shown in the Fig. 2b).

1
2 2
3 6 3
4 12 12 4
5 20 30 20 5
6 30 60 60 30 6
....
a)

1
3 3
6 12 6
10 30 30 10
15 60 90 60 15
21 105 210 210 105 21
....
b)

Fig. 2

Compare the base layer with the second layer.
We see that the second layer is obtained by multiplying the rows of Pascal's triangle by 1, 2, 3, 4, 5, ... Can you find this sequence in Pascal's triangle?
Compare the base layer with the third layer.
We see that the third layer is obtained by multiplying the rows of Pascal's triangle by 1, 3, 6, 10, 15, ... Can you find this sequence in Pascal's triangle?
Generalize it. That is, answer the question „what is the number triangle for the n-th layer?"

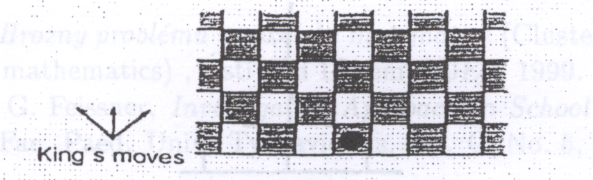

Fig. 3

Investigation of paths on a chessboard

P_2: *The chessboard in* Fig. 3 *is bounded on the bottom only. A king* is placed on a square in the bottom and can only move in three directions - up, up-left and up-right. For each square on Fig. 3 find the number of paths the king may take to this square from the start square, and write this number in the square.

Solution of P_2:

The strategy of going step by step. The strategy of going step-by-step leads to number triangle in Fig. 4. Investigate this triangle. Can you discover there, for example, any Tribonacci sequence[2]?

1	3	6	7	6	3	1
	1	2	3	2	1	
		1	1	1		
			1			

Fig.4

In conclusion, we wish to show an interesting relationship between moves of a king on an „infinite chessboard" and a particular set of moves on a cubic grid.

[2]The first three terms of the sequence can be chosen arbitrarily, and each succeeding term is the sum of three terms preceding it. The Tribonacci sequence we examine here is: 1, 1, 2, 4, 7, 13, 24, ...

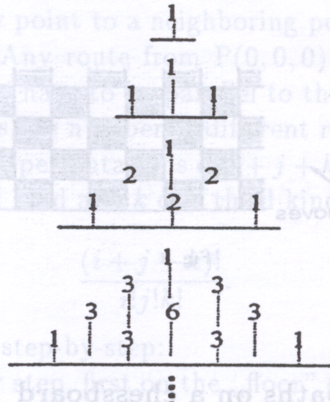

Fig. 5

Cut the cubic grid by planes

Cut the cubic grid (see Fig. 1) by planes π_n through the points $A_n(n,0,0)$, $B_n(0,n,0)$, $C_n(0,0,n)$ for $n = 0,1,2,3,\ldots$. The labels lying on π_n create a number triangle, forming the n-th layer of the pyramid (see Fig. 5).

Now compare the number triangles in Fig. 5 with number triangle in Fig. 4. If you add the numbers in each triangle connected by the vertical lines and then use the sums as rows in a new number triangle, you get the following:

1

1, 1, 1

1, 2, 3, 2, 1

1, 3, 6, 7, 6, 3, 1

⋮

which is just the triangle in Figure 4!

A particularly challenging investigation would be to explain what was shown here.

References

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[2] L. Frobisher, *Insights into Teaching Mathematics*, London, University of Leeds, 1996.

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