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Teaching Calculus with Original Historical Sources - Γ Function

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Teaching calculus with original historical sources has an important advantage. It is convenient to observe, how a new mathematical idea grows up in the mind of the author and it is convenient to make a similar image in the mind of students. It is possible to watch causes, which lead to the creations of the term. It is very important too, that students understand the way of thinking of the author, and the students can master work methods, which are used by creative mathematicians.

In the following text I would like to talk about the historical development of the function Γ. I mention only some aspects, because this paper is too short for a detailed description. I choose this topic, because I have an auspicious reference from students of PF UJEP, who were visiting my lectures last year.

One of the simplest task, which one can meet with work on number sequences, is looking for a general expression which would give all the terms of the progression

$$1,\ 1+2,\ 1+2+3,\ 1+2+3+4,\ \ldots \tag{1}$$

If we denote by S_n the n-th term of the progression (1), then $S_n = \frac{1}{2}n(n+1)$ holds. The meaning of the rule will become obvious for the famous task of adding of the first hundred positive integers. Instead of slow adding it simplifies computation to three fixed arithmetic operations: one addition, one multiplication and one division. Moreover, the formula solves the problem of interpolating between the terms of progression (1), because we obtain the meaningful results by substituing of noninteger values into the formula. Function Γ arose from as a result of finding similar formula for the sequence of factorials. Mathematicians were looking for a formula, which would enable to compute values of factorials without a tedious multiplication and, moreover, which would solve a problem of interpolating between the factorials. The difficulties turned up for a very rapidly growth of factorials. The number 100! has 158 digits if it is written out.

A lot of mathematicians such as JAMES STIRLING (1692 – 1770), CHRISTIAN GOLBACH (1690 – 1764) or DANIEL BERNOULLI (1700 – 1784), were interested in a solving of the problem mentioned above. But the first mathematician who established the theory of the function Γ was surely LEONHARD EULER (1707 – 1783).

If we want to solve the described problem at the present time, we have an easy target. Our solving consists in constructing such a function, which is equal 1 for points 0 and 1, acquired an arbitrary values between the ones. The values of the rest variable could be computed by help of the recurrence relationship $(n+1)! = (n+1) \cdot n!$

This approach is based on today's concept of a function. Today, a function is a relationship between two sets of numbers, where an element of the first set is assigned to an element of the second set. But in the 18th century, a function meant such an *expressio analytica*, i.e., formula which could be derived from elementary manipulations with addition, subtraction, multiplication, division, differentiation, integration, etc. Euler's task was to find an analytical expression which would yield factorials when a positive integer was inserted, but which would still be meaningful for other values of the variable. Now let's show how Euler solved the mentioned problem.

Euler's solution is described in [2]. The English translation [3] of [2] is available in the URL `http://home.sandiego.edu/~langton/`. At the beginning of the article Euler expressed factorials in the form of an infinite product

$$n! = \frac{1 \cdot 2^n}{1+n} \cdot \frac{2^{1-n} \cdot 3^n}{2+n} \cdot \frac{3^{1-n} \cdot 4^n}{3+n} \cdot \frac{4^{1-n} \cdot 5^n}{4+n} \cdots \tag{2}$$

This assertion is introduced in [2] without an explanation of derivation. Later Euler described his solution in the article *De termino generali serierum hypergeometricarum* (1776).

As Euler wrote, this product never breaks off whether n is a whole number or a fraction, but only gives approximately values, except in the cases $n = 0$ and $n = 1$, in which it just becomes 1. There is a question in this place, how far Euler was able to make the limit transformation of (2) in his mind.

Euler knew that (2) is not too applicable for a computation. But right side (2) is defined for all kinds of n other than negative integers. Euler noticed that if we set $n = 1/2$ then (2) becomes the famous infinite product of JOHN WALLIS (1616 – 1703)

$$\sqrt{\frac{2 \cdot 4}{3 \cdot 3} \frac{4 \cdot 6}{5 \cdot 5} \frac{6 \cdot 8}{7 \cdot 7} \frac{8 \cdot 10}{9 \cdot 9} \cdots} \tag{3}$$

This formula meant for Wallis that the circle was to the square of its diameter as

$$2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdots \quad \text{to} \quad 3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdots$$

If diameter of the circle is equal 1, the area of one will be

$$\frac{2\cdot4}{3\cdot3}\cdot\frac{4\cdot6}{5\cdot5}\cdot\frac{6\cdot8}{7\cdot7}\cdot\frac{8\cdot10}{9\cdot9}\cdots$$

From this we can conclude that for $n = 1/2$ the product (2) is equal to the square root of the circle with diameter $= 1$. The correspondence between the factorials and the area of circle led Euler to the idea to express factorials by integral.

Euler took up the integral $\int x^e(1-x)^n dx$. First Euler expressed

$$(1-x)^n = 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot2}x^2 - \frac{n(n-1)(n-2)}{1\cdot2\cdot3}x^3 + \cdots, \tag{4}$$

then he multiplied both sides of the equation by x^e and integrated it. Thus he obtained

$$\int x^e(1-x)^n\,dx =$$

$$= \frac{x^{e+1}}{e+1} - \frac{nx^{e+2}}{1\cdot(e+2)} + \frac{n(n-1)x^{e+3}}{1\cdot2\cdot(e+3)} - \frac{n(n-1)(n-2)x^{e+4}}{1\cdot2\cdot3\cdot(e+4)} + \cdots. \tag{5}$$

The value of form on the right side of (5) is equal 0 for $x = 0$. Euler set $x = 1$, thereby the indefinite integral in (5) becomes a definite integral $\int_0^1 x^e(1-x)^n dx$ (I will resign Euler's notation and I will write $\int$ instead of $\int_0$). Thus, Euler obtained

$$\frac{1}{e+1} - \frac{n}{1\cdot(e+2)} + \frac{n(n-1)}{1\cdot2\cdot(e+3)} - \frac{n(n-1)(n-2)}{1\cdot2\cdot3\cdot(e+4)} + \cdots. \tag{6}$$

This exppresion has an important role, because it yields the following values for given n

n	0	1	2	3
a_n	$\frac{1}{e+1}$	$\frac{1}{(e+1)(e+2)}$	$\frac{1\cdot2}{(e+1)(e+2)(e+3)}$	$\frac{1\cdot2\cdot3}{(e+1)(e+2)(e+3)(e+4)}$

The rule which provides the next terms is obvious. Thus, he obtained a general formula for non-negative integers and arbitrary e

$$\int x^e(1-x)^n dx = \frac{1\cdot2\cdot3\cdot4\cdots n}{(e+1)(e+2)\cdots(e+n+1)}. \tag{7}$$

Multiplying both sides of (7) by term $e+n+1$ led to equality

$$(e+n+1)\int x^e(1-x)^n dx = \frac{1\cdot2\cdot3\cdots n}{(e+1)(e+2)\cdots(e+n)}. \tag{8}$$

Substituing e by fraction f/g and dividing by g^n changed (8) into a form

$$\frac{(f+(n+1)g}{g^{n+1}}\int x^{f/g}(1-x)^n dx = \frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)\cdots(f+ng)}. \quad (9)$$

Setting $f=1$ and $g=0$ led to

$$1\cdot 2\cdot 3\cdot\ldots\cdot n = \int \frac{x^{\frac{1}{0}}(1-x)^n dx}{0^{\frac{1}{0}}} \quad (10)$$

In the next part of the article Euler investigated the sense of the form on the right side of (10). Therefore, he put $x^{\frac{g}{f+g}}$ in place of x, $\frac{g}{f+g}x^{\frac{-f}{g+f}}dx$ in place of dx and set $f=1$ and $g=0$. Thus, he obtained

$$\frac{f+(n+1)g}{g^{n+1}}\int\frac{g}{f+g}\left(1-x^{\frac{g}{f+g}}\right)^n dx, \quad \text{resp.} \quad \int\frac{(1-x^0)^n}{0^n}dx.$$

Now Euler considered the related expression $(1-x^z)/z$ for vanishing z. He differentiated the numerator and denumerator by a known (Hôspital's) rule and obtained

$$\frac{-x^z dz\ lx}{dz}, \quad (11)$$

where lx means $\ln x$ in a present time. The term (11) produced $-lx$ for $z=0$. Thus, $(1-x^0)/0 = -lx$ and $(1-x^0)^n/0^n = (-lx)^n$. From this reason Euler derived that integral $\int\frac{(1-x^0)^n}{0^n}dx$ can be replaced by integral $\int(-lx)^n dx$. Consequently the general term of the sequence factorials is $\int(-lx)^n dx$. Thus

$$n! = \int_0^1 (-\ln x)^n\, dx. \quad (12)$$

The recurrence relation

$$\int_0^1(-\ln x)^n dx = [x(-\ln x)^n]_0^1 + n\int_0^1(-\ln x)^{n-1}dx = n\int_0^1(-\ln x)^{n-1}dx$$

shows, that (12) really produced factorials. Moreover, the integral (12) makes a meaningful values for arbitrary $n\in(-1,\infty)$. By these means Euler solved the problem of the interpolation between the factorials for arbitrary positive real number.

Substituing e^{-t} for x in (12) we obtain a new relation

$$n! = \int_0^\infty e^{-t}t^n dt. \quad (13)$$

This integral is defined for all $n \in (-1, \infty)$. Later ADRIEN MARIE LEGENDRE (1752 – 1833) started to work with (13). He modified it into a current form

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. \tag{14}$$

It is called the second Eulerian integral, or the gamma function and is denoted by a letter Γ. The function Γ is defined by (14) for all $x \in (0, \infty)$ and its relation to factorials is given by the formula $\Gamma(n+1) = n!$.

A considerable advance came with the paper [4]. CARL FRIEDRICH GAUSS (1777 – 1855) solved the problem of interpolation between the factorials by introducting the function

$$\Pi(k, z) = \frac{1 \cdot 2 \cdot 3 \cdots k}{(z+1)(z+2)(z+3)\cdots(z+k)} k^z. \tag{15}$$

In [4] Gauss derived important results, i.e. famous *multiplication formula*:

$$\frac{n^{nz}\Pi z \cdot \Pi\left(z - \frac{1}{n}\right) \Pi\left(z - \frac{2}{n}\right) \cdots \Pi\left(z - \frac{n-1}{n}\right)}{\Pi nz} = \frac{(2\pi)^{\frac{1}{2}(n-1)}}{\sqrt{n}}.$$

Later Gauss proved that (15) can be used for extension factorials to the region $D := C \setminus \{-n; n \in N\}$, where C denotes the set of complex numbers and N stands for the set of positive integers.

In the next years many mathematicians started to looking for the conditions, which characterized the function Γ. The functional equation $f(x+1) = xf(x)$ together with the condition $f(1) = 1$ were a natural requirements, but it was showed that these conditions aren't so strong for unique characterization of the function Γ. It was necessary to look for additional conditions. It graduelly showed, that requirements on continuity, differentiability, convexity aren't sufficient. The logarithmic convexity of the function Γ is a searched property.

HARALD BOHR (1887–1951) and JOHANNES MOLLERUP (1872–1937) proved that Γ-function is the only function that satisfies $f(x+1) = xf(x)$, $f(1) = 1$ and $\log f(x)$ is a convex function. The original proof was considerably simplified by EMIL ARTIN (1898–1962) in [1] and so the theorem is now known as *Bohr-Mollerup-Artin theorem*:

Theorem. *Let f be a function with the following properties:*

(a) $f(x+1) = xf(x), \quad x \in R_+$

(b) $f(1) = 1$

(c) $f(x)$ is log-convex function on R_+

Then $f(x) = \Gamma(x)$ *for all* $x \in R_+$ *where* $R_+ = (0; \infty)$.

Artin showed that the Bohr-Mollerup-Artin's theorem considerably helps to obtain simple proofs of some well-known assertions. Let us show the importance of this theorem: We will prove a well-known relation between Γ-function and the Beta function.

Euler studied another integral connected with Γ-function. Now we call it Euler integral "of the first kind" or Beta function. It is defined by equation:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t. \tag{16}$$

Its values are dependent on both variables x and y. This improper integral exists for all $x > 0$ and $y > 0$. The main result of the theory of Beta function is the so called Euler's identity:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y \in R_+. \tag{17}$$

The idea of proof consists in verifying the conditions of the Bohr-Mollerup-Artin's theorem. Substituing $x+1$ for x and using (16), we obtain

$$B(x+1, y) = \int_0^1 t^x(1-t)^{y-1}\mathrm{d}t = \int_0^1 (1-t)^{x+y-1}\left(\frac{t}{1-t}\right)^x \mathrm{d}t.$$

Using integration by parts we derive the equation:

$$B(x+1, y) = \frac{x}{x+y}B(x, y) \tag{18}$$

which holds for all $x, y \in R_+$. Now, we will study function (16) as a function of variable x with fixed value of y. We introduce a function $f(x)$ to obtain function which is a solution of equation $f(x+1) = xf(x)$:

$$f(x) = B(x, y)\Gamma(x+y). \tag{19}$$

Let's verify that this function satisfies all the conditions of the Bohr-Mollerup-Artin's theorem.

The condition $f(x+1) = xf(x)$ is easily seen from

$$f(x+1) = B(x+1, y)\Gamma(x+y+1) = \frac{x}{x+y}B(x, y)(x+y)\Gamma(x+y) = xf(x).$$

Clearly, the function $f(x)$ is log-convex, as a product of two log-convex function. The condition $f(1) = 1$ is not true since

$$B(1,y) = \int_0^1 (1-t)^{y-1}\mathrm{d}t = \frac{1}{y}, \quad \text{and also} \quad f(1) = \frac{1}{y}\Gamma(1+y) = \Gamma(y).$$

Therefore, $f(1) = B(1,y)\Gamma(1+y) = y^{-1}y\,\Gamma(y) = \Gamma(y)$. If the function $f(x)$ satisfies conditions (a), (c), it is easy to realize (b). Since we suppose (c) we have $f(1) > 0$. Then $f(x)/f(1)$ satisfies all three conditions and therefore it is the Γ-function by Bohr-Mollerup-Artin theorem. Now we have

$$f(x) = f(1)\cdot\Gamma(x) \tag{20}$$

and for our function $f(x)$ from (20) we have

$$f(x) = \Gamma(y)\Gamma(x). \tag{21}$$

Comparison of (19) with (21) gives the equation (17) for integral in (16).

Preparation of teaching, which is led by this way, needs a detailed survey about the development of mathematics. It is good to know sources providing original historical papers. I have used two main sources for a preparation for my lecture. The first one is the Library of MFF UK in Prague. I found the second one in Göttinger Digitalisierung Zentrum (GDZ), which is possible to visit in the website:
`http://gdz.sub.uni-goettingen.de/en/index.html`.
Other informations about teaching with original historical sources are available in `http://math.nmsu.edu/~history/`.

References

[1] E. Artin, *Einführung in die Theorie der Gammafunktion.* Leipzig, 1931.

[2] L. Euler, *De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt.* Opera Omnia, vol. I_{14}, Leipzig-Berlin, 1924.

[3] L. Euler, *On transcendental progression that is, those whose general terms cannot be given algebraically.* Comentarii academiae scientiarum Petropolitanae 5 (1730/1). Translated by S.G.Langton, University of San Diego, 1999.

[4] C. F. Gauss, *Allgemeine Untersuchungen über die unendliche Reihe.* Springer, Berlin 1888.

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