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Multi-valued n–sequential propositional logic

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Introduction. The method of sequents for the classical logical calculi was first introduced by G. Gentzen [5] to formalize the concept of proof occurring in the deductive practice. To some extent, the so-called tableau method or method of dyadic trees is dual to Gentzen's sequential method. The tableau method was independently introduced by E. Beth [2], J. Hintikka [6] and K. Schütte [15]. In the sequent method of logical calculi construction, the main notion is that of a sequent. A sequent is defined as either a pair or a finite sequence of finite sets of formulae. The essence of the method of dyadic trees is proving that some formulae are tautologies by refuting their negations.

The method of sequents for n–valued logics was developed by V. G. Kirin [7], G. Rousseau [11], [12], [13] and Z. Saloni [14], [15],. In the above papers, sequents are defined and interpreted in a way similar to that used in two–valued logic. In these constructions of logical calculi, a number of unary logical connectives and some constants are necessary. The method of trees, or tableau method for n–valued propositional calculi, was introduced by S. J. Surma [20], [21] and W. Suchoń [19]. Our method of n–sequents requires neither the application of designated formulae nor additional unary logical connectives or logical constants. The purpose of this paper is to present the so-called n–sequent method for the n–sequential logics, which is dual to this method is the notion of n–sequent, which is similar to the notion of sequent from [7], [11] and [15], but is interpreted in a different way.

1. Algebraic preliminaries

Let $\Omega = (\Omega_m | m \in N)$ be a family of sets of operations (cf Cohn [4]).

By an Ω–algebra we mean an ordered pair $\mathcal{A} = (A, op)$, where A is a set, and $op = \left(op_m : \Omega_m \longrightarrow A^{(A^m)} | m \in N\right)$ is a family of functions, with $A^{(A^m)}$ denoting the set of all functions $f : A^m \longrightarrow A$.

Let V be a denumerable set of propositional variables. We define the valuation of Ω–terms over V in the Ω–algebra $\mathcal{A} = (A, (op_m | m \in N))$ to be the unique function $\| \bullet \|_{\mathcal{A}} : \Omega^S \longrightarrow A$ satisfying the following conditions:

$$\|x\|_{\mathcal{A}}(v) = v(x) \ \text{ for every } \ v \in A^V, \ x \in V$$

$$\|\omega(t_1, \ldots, t_m)\|_{\mathcal{A}}(v) = op_m(\omega)(\|t_1\|_{\mathcal{A}}(v), \ldots, \|t_m\|_{\mathcal{A}}(v))$$
$$\text{for all } \ \omega \in \Omega_m, t_1, \ldots, t_m \in \Omega^S, m \in N, v \in A^V.$$

2. Propositional S – calculus (semantic part)

Let $\mathbf{E} = \{1, 2, \ldots, n\}$. By a specification we mean an ordered triple

$$S = (\mathbf{E}, D, C),$$

where $D \subseteq \mathbf{E}$ is a set of distinguished elements, and

$$C \subseteq \bigcup_{m \in N} \mathbf{E}^{\mathbf{E}^m}.$$

By the type determined by a specification $S = (\mathbf{E}, D, C)$ we mean the family $\Omega^S = (\Omega^S_m | m \in N)$, where $\Omega^S_m = C \cap \mathbf{E}^{\mathbf{E}^m}$ for every $m \in N$.

By the matrix algebra associated with a specification $S = (\mathbf{E}, D, C)$ we mean the Ω^S– algebra

$$\mathcal{A}_S = (\mathbf{E}, (op_m | m \in N)) \ \text{ such that } op_m(f) = f$$
$$\text{for every } \ f \in \Omega^S_m, m \geq 0.$$

By a propositional calculus (more precisely, a propositional S–calculus) we mean an ordered quadruple

$$P_C = (S, L, \| \bullet \|(\bullet), Pr), \ \text{ where}$$

(i) $S = (\mathbf{E}, D, C)$ is a specification,

(ii) L, called the language of P_C, is an ordered pair (V, F_S) such that V is a denumerable set of variables, and F_S is a set of Ω^S –terms called the set of formulae,

(iii) $\| \bullet \| : \mathbf{E}^V \longrightarrow \mathbf{E}$ is a valuation of Ω – terms in the matrix algebra associated to S.

(iv) The data contained in Pr will be defined in the next section. Pr contains information about the mechanism for acceptance or refutation of formulae.

Let $P_C = (S, L, \| \bullet \|, Pr)$ be a propositional S–calculus, where $S = (\mathbf{E}, D, C)$ and $L = (V, F_S)$.

By a semantic tautology of the propositional S–calculus we mean a formula $a \in F_S$ such, that for every $v \in \mathbf{E}^V$,

$$\|a\|(v) \in D$$

3. Propositional calculus (formal part)

Let $\Gamma_1, \ldots, \Gamma_n$ be finite sets of formulae, i.e. let the elements of Γ_i be in F_S for $i = 1, \ldots, n$. Some of them may be empty. A sequent is an ordered n–tuple $(\Gamma_1, \ldots, \Gamma_n)$, which we denote by $\Gamma_1 \vdash \ldots \vdash \Gamma_n$. We will also write Γ_1, α, instead of $\Gamma_i \cup \{\alpha\}$. Sequents will be denoted by Σ, with indices if necessary.

By an overfilled sequent we mean a sequent $\Sigma = \Gamma_1 \vdash \ldots \vdash \Gamma_n$ such that either $\Gamma_j \cap \Gamma_k \neq \emptyset$ for some j, k in $\{1, \ldots, n\}$, $j \neq k$, or sequents $\Gamma_1 \vdash \ldots \vdash \Gamma_i,\ \omega_{t_i}(\alpha_1, \ldots, \alpha_{m_i}) \vdash \ldots \vdash \Gamma_n$ for $i = 1, 2, \ldots, n$ such that there does not exist valuation $v : V \to E_n$ for which the equation $\|\omega_{t_1}(\alpha_1, \ldots, \alpha_{m_i})\|(v) = i$ is satisfied for any $\omega_{t_i} \in \Omega_{m_i}$ and any formulae $\alpha_1, \alpha_2, \ldots, \alpha_{m_i} \in F_S$. To present the rules given in the sequel in a concise form, we shall adopt the following notation.

Let $\Sigma = \Gamma_1 \vdash \ldots \vdash \Gamma_n$ be a sequent and let $\alpha_1, \ldots, \alpha_m$ be an Ω^S – term, $x \in \mathbf{E}^m$, and $\omega \in \Omega^S_m$. Moreover, let $\Sigma_x = \Gamma'_1 \vdash \ldots \vdash \Gamma'_n$ be a sequent such that

$$\Gamma'_i = \Gamma_i \cup \{\alpha_k \mid pr_k(x) = i\}$$

Then the schema of the rule for introducing a connective ω into the sequen Γ_j of the sequent $\Sigma = \Gamma_1 \vdash \ldots \vdash \Gamma_n$ is as follows:

$$(r) \quad \frac{\{\Sigma_x \mid x \in \mathbf{E}^m \ \&\ (op^S_m(\omega))(x) = j\}}{\Gamma_1 \vdash \ldots \vdash \Gamma_{j-1} \vdash \Gamma_j, \omega(\alpha_1, \ldots, \alpha_m) \vdash \Gamma_{j+1} \vdash \ldots \vdash \Gamma_n}$$

for $j = 1, \ldots, n$

By an unordered tree we shall mean a collection $T = (D, D', l, R, x_1)$ such that:

(1) D is a set of elements called points.

(1′) D' is a subset of D.

(2) l is a function which assigns to each point $x \in D$ a positive integer $l(x)$, called the level of x.

(3) R is a relation in D (i.e. $R \subseteq D^2$); we read xRy as „x is a predecessor of y" or „y is the successor of x".

This relation must satisfy the following conditions:

(i) There is a unique point x_1 of lewel 1. This point is called the root of the tree.

(ii) Every point $x \in D, x \neq x_1$, has a unique successor.

(iii) For any points x, y, if y is a successor of x, then $l(y) = l(x) + 1$.

(iv) $D' = \{y \mid \{x \mid xRy\} = \emptyset\}$.

(v) $c(y) = card\ \{x \mid xRy\} \leq n^m$.

By a proof tree in the sequential calculus we mean a tree $T = (P, P', l, R\alpha_0)$, where

(1) P is a set of sequents.

(2) $P' \subseteq P$ is a set of overfilled sequents.

(3) R is a relation on P defined by
$\Sigma_1 R \Sigma_2 \Longleftrightarrow$ there exists a rule $r \in (r)$ such that
Σ_2 is the conclusion of r, and Σ_1 is one of its premises.

A sequent $\Sigma = \Gamma_1 \vdash \ldots \vdash \Gamma_n$ is a terminal sequent if there exists a formula $\alpha \in F_S$ and $j, 1 \leq j \leq n$, such that $\Gamma_i = \emptyset$ for every $i \neq j, 1 \leq i \leq n$, and $\Gamma_j = \{\alpha\}$.

A propositional formula α is a theorem in the n–sequential logic if and only if there exist sets of overfilled sequents, from which the following terminal sequents are provable:

$$\Sigma_1 = \Gamma_{11} \vdash \Gamma_{12} \vdash \ldots \vdash \Gamma_{1,r-1} \vdash \Gamma_{1,r} \vdash \Gamma_{1,r+1} \vdash \ldots \vdash \Gamma_{1n}$$

$$\Sigma_2 = \Gamma_{21} \vdash \Gamma_{22} \vdash \ldots \vdash \Gamma_{2,r-1} \vdash_{2,r} \vdash \Gamma_{2,r+1} \vdash \ldots \vdash \Gamma_{2n}$$

....

$$\Sigma_{r-1} = \Gamma_{r-1,1} \vdash \Gamma_{r-1,2} \vdash \ldots \vdash \Gamma_{r01,r-1} \vdash \Gamma_{r-1,r} \vdash \Gamma_{r-1,r+1} \vdash \ldots \Gamma_{r-1,n}$$

where

$$\Gamma_{ij} = \begin{cases} \{\alpha\} & \text{if} \quad i = j, \\ \emptyset & \text{if} \quad i \neq j, \end{cases}$$

for $i = 1, \ldots, r-1$ and $j = 1, \ldots, n$, and r is the smallest designated valued of D.

Examples:

1. Reichenbach's three-sequential propositional calculus.

Let $C_1 = \{\sim, -, \neg\}$ be a set of unary connectives, and let $C_2 = \{\vee, \wedge, \Rightarrow, \supset, \rightarrow, \equiv, \Leftrightarrow\}$ be a set of binary ones. The set of formulas for Reichenbach's three-sequential propositional calculus over V, C_1 and C_2 is defined in the standard way. Reichenbach's three-valued calculus consists of rules applied to either overfilled sequents, or sequents obtained earlier, where by overfilled sequents we understand here sequents $\Gamma_1 \vdash \Gamma_2 \vdash \Gamma_3$ such that $\Gamma_i \cap \Gamma_j \neq \emptyset$, for $i \neq j$, $i, j = 1,2,3$, and sequents $\Gamma_1, \neg\alpha \vdash \Gamma_2 \vdash \Gamma_3, \Gamma_1 \vdash \Gamma_2, \alpha \supset \beta \vdash \Gamma_3, \Gamma_1 \vdash \Gamma_2, \alpha \Leftrightarrow \beta \vdash \Gamma_3$, for any $\alpha, \beta \in F_2$ and any $\Gamma_1, \Gamma_2, \Gamma_3 \subseteq F_S$.

The rules for introducing connectives into the sequens Σ_i of the sequents $\Sigma = \Gamma_1 \vdash \Gamma_2 \vdash \Gamma_3$ are as follows:

$$(r1 \sim) \quad \frac{\Gamma_1 \vdash \Gamma_2 \vdash \Gamma_3, \alpha}{\Gamma_1, \sim \alpha \vdash \Gamma_2 \vdash \Gamma_3},$$

$$(r2 \sim) \quad \frac{\Gamma_1, \alpha \vdash \Gamma_2 \vdash \Gamma_3}{\Gamma_2 \vdash \Gamma_2, \sim \alpha \vdash \Gamma_3},$$

$$(r3 \sim) \quad \frac{\Gamma_1 \vdash \Gamma_2, \alpha \vdash \Gamma_3}{\Gamma_1 \vdash \Gamma_2 \vdash \Gamma_3, \sim \alpha},$$

$$(r1-) \quad \frac{\Gamma_1 \vdash \Gamma_2 \vdash \Gamma_3, \alpha}{\Gamma_1, -\alpha \vdash \Gamma_2 \vdash \Gamma_3},$$

$$(r2-) \quad \frac{\Gamma_1 \vdash \Gamma_2, \alpha \vdash \Gamma_3}{\Gamma_1 \vdash \Gamma_2, -\alpha \vdash \Gamma_3},$$

$$(r3-) \quad \frac{\Gamma_1, \alpha \vdash \Gamma_2 \vdash \Gamma_3}{\Gamma_1 \vdash \Gamma_2 \vdash \Gamma_3, -\alpha},$$

$$(r2\neg) \quad \frac{\Gamma_1 \vdash \Gamma_2 \vdash \Gamma_3, \alpha}{\Gamma_1 \vdash \Gamma_2, \neg \vdash \Gamma_3},$$

$$(r3\neg) \quad \frac{\Gamma_1, \alpha \vdash \Gamma_2 \vdash \Gamma_3;\ \Gamma_1 \vdash \Gamma_2, \alpha \vdash \Gamma_3}{\Gamma_1 \vdash \Gamma_2 \vdash \Gamma_3, \neg\alpha},$$

$$(r1\vee) \quad \frac{\Gamma_1, \alpha, \beta \vdash \Gamma_2 \vdash \Gamma_3}{\Gamma_1, \alpha \vee \beta \vdash \Gamma_2 \vdash \Gamma_3},$$

$$(r2\vee) \quad \frac{\Gamma_1, \alpha \vdash \Gamma_2, \beta \vdash \Gamma_3;\ \Gamma_1, \beta \vdash \Gamma_2, \alpha \vdash \Gamma_3;\ \Gamma_1 \vdash \Gamma_2, \alpha, \beta \vdash \Gamma_3}{\Gamma_1 \vdash \Gamma_2, \alpha \vee \beta \vdash \Gamma_3},$$

$$(r3\vee) \quad \frac{\Gamma_1 \vdash \Gamma_2 \vdash \Gamma_3, \alpha;\ \Gamma_1 \vdash \Gamma_2 \vdash \Gamma_3, \alpha}{\Gamma_1 \vdash \Gamma_2 \vdash \Gamma_3, \alpha \vee \beta},$$

$$(r1\wedge)\quad \frac{\Gamma_1,\alpha\vdash\Gamma_2\vdash\Gamma_3;\quad \Gamma_1,\beta\vdash\Gamma_2\vdash\Gamma_3}{\Gamma_1,\alpha\wedge\beta\vdash\Gamma_2\vdash\Gamma_3},$$

$$(r2\wedge)\quad \frac{\Gamma_1\vdash\Gamma_2,\alpha,\beta\vdash\Gamma_3;\ \Gamma_1\vdash\Gamma_2,\alpha\vdash\Gamma_3,\beta;\ \Gamma_1\vdash\Gamma_2,\beta\vdash\Gamma_3,\alpha}{\Gamma_1\vdash\Gamma_2,\alpha\wedge\beta\vdash\Gamma_3},$$

$$(r3\wedge)\quad \frac{\Gamma_1\vdash\Gamma_2\vdash\Gamma_3,\alpha,\beta}{\Gamma_1\vdash\Gamma_2\vdash\Gamma_3,\alpha\wedge\beta},$$

$$(r1\Rightarrow)\quad \frac{\Gamma_1,\beta\vdash\Gamma_2\vdash\Gamma_3,\alpha}{\Gamma_1,\alpha\Rightarrow\beta\vdash\Gamma_2\vdash\Gamma_3},$$

$$(r2\Rightarrow)\quad \frac{\Gamma_1,\beta\vdash\Gamma_2,\alpha\vdash\Gamma_3;\ \gamma_1\vdash\Gamma_2,\beta\vdash\Gamma_3,\alpha}{\Gamma_1\vdash\Gamma_2,\alpha\Rightarrow\beta\vdash\Gamma_3},$$

$$(r3\Rightarrow)\quad \frac{\Gamma_1\vdash\Gamma_2,\vdash\beta\vdash\Gamma_3;\Gamma_1\vdash\alpha\Gamma_3\alpha;\Gamma_1\vdash\Gamma_2\vdash\Gamma_3,\beta}{\Gamma_1\vdash\Gamma_2\vdash\Gamma_3,\alpha\Rightarrow\beta},$$

$$(r1\supset)\quad \frac{\Gamma_1,\beta\vdash\Gamma_2\vdash\Gamma_3,\alpha;\ \Gamma_1\vdash\Gamma_2,\beta\vdash\Gamma_3,\alpha}{\Gamma_1,\alpha\supset\beta\vdash\Gamma_2\vdash\Gamma_3},$$

$$(r3\supset)\quad \frac{\Gamma_1,\alpha\vdash\Gamma_2\vdash\Gamma_3;\Gamma_1\vdash\Gamma_2\vdash\Gamma_3,\beta}{\Gamma_1\vdash\Gamma_2\vdash\Gamma_3,\alpha\supset\beta},$$

$$(r1\to)\quad \frac{\Gamma_1,\beta\vdash\Gamma_2\vdash\Gamma_3,\alpha}{\Gamma_1,\alpha\to\beta\vdash\Gamma_2\vdash\Gamma_3},$$

$$(r2\to)\quad \frac{\Gamma_1,\alpha\vdash\Gamma_2\vdash\Gamma_3;\quad \Gamma_1\vdash\Gamma_2,\alpha\vdash\Gamma_3}{\Gamma_1\vdash\Gamma_2,\alpha\to\beta\vdash\Gamma_3},$$

$$(r3\to)\quad \frac{\Gamma_1\vdash\Gamma_2\vdash\Gamma_3,\alpha,\beta}{\Gamma_1\vdash\Gamma_2\vdash\Gamma_3,\alpha\to\beta},$$

$$(r1\equiv)\quad \frac{\Gamma_1,\alpha\vdash\Gamma_2\vdash\Gamma_3,\beta;\ \Gamma_1,\beta\vdash\Gamma_2\vdash\Gamma_3,\alpha}{\Gamma_1,\alpha\equiv\beta\vdash\Gamma_2\vdash\Gamma_3},$$

$$(r2\equiv)\quad \frac{\Gamma_1,\alpha\vdash\Gamma_2,\beta\vdash\Gamma_3;\ \Gamma_1,\beta\vdash\Gamma_2,\alpha\vdash\Gamma_3;\ \Gamma_1\vdash\Gamma_2,\alpha\vdash\Gamma_3,\beta;\ \Gamma_1\vdash\Gamma_2,\beta\vdash\Gamma_3}{\Gamma_1\vdash\Gamma_2,\alpha\equiv\beta\vdash\Gamma_3}$$

$$(r3\equiv)\quad \frac{\Gamma_1,\alpha,\beta\vdash\Gamma_2\vdash\Gamma_3;\ \Gamma_1\vdash\Gamma_2,\alpha,\beta\vdash\Gamma_3;\ \gamma_1\vdash\gamma_2\vdash\Gamma_3,\alpha,\beta}{\Gamma_1\vdash\Gamma_2\vdash\Gamma_3,\alpha\equiv\beta},$$

$$(r1\Leftrightarrow)\quad \frac{\Gamma_1,\alpha\vdash\Gamma_2,\beta\vdash\Gamma_3;\ \Gamma_1,\alpha\vdash\Gamma_2\vdash\Gamma_3,\beta;\Gamma_1,\beta\vdash\Gamma_2,\alpha\vdash\Gamma_3;\quad \Gamma_1\vdash\Gamma_2,\alpha\vdash\Gamma_3,\beta;\ \Gamma_1,\beta\vdash\Gamma_2\vdash\Gamma_3,\alpha;\Gamma_1\vdash\Gamma_2,\beta\vdash\Gamma_3,\alpha}{\Gamma_1,\alpha\Leftrightarrow\beta\vdash\Gamma_2\vdash\Gamma_3},$$

$$(r3 \Leftrightarrow) \quad \frac{\Gamma_1, \alpha, \beta \vdash \Gamma_2 \vdash \Gamma_3;\ \Gamma_1 \vdash \Gamma_2, \alpha, \beta \vdash \Gamma_3;\ \Gamma_1 \vdash \Gamma_2 \vdash \Gamma_3, \alpha, \beta}{\Gamma_1 \vdash \Gamma_2 \vdash \Gamma_3, \alpha \Leftrightarrow \beta}$$

Now we shall show that the formula $\alpha \equiv - - \alpha$ is a theorem in Reichenbach's three–sequential propositional calculus. There exist two sets of overfilled sequents:

$$S_1 = \{\alpha \vdash \emptyset \alpha\} \qquad S_2 = \{\alpha \vdash \alpha \vdash \emptyset, \emptyset \vdash \alpha \vdash \alpha\}$$

from which we get the following proofs:

$\alpha \vdash \emptyset \vdash \alpha$ — rule (r1–) — $\alpha, -\alpha \vdash \emptyset \vdash \emptyset$ — rule (r3–) — $\alpha \vdash \emptyset \vdash - - \alpha$

$\alpha \vdash \emptyset \vdash \alpha$ — rule (r3–) — $\emptyset \vdash \emptyset \vdash \alpha, -\alpha$ — rule (r1–) — $- -\alpha \vdash \emptyset \vdash \alpha$

rule (r1 $\equiv$)

$$\alpha \equiv - - \alpha \vdash \emptyset \vdash \emptyset$$

$\alpha \vdash \alpha \vdash \emptyset$	$\alpha \vdash \alpha \vdash \emptyset$	$\emptyset \vdash \alpha \vdash \alpha$	$\emptyset \vdash \alpha \vdash \alpha$
rule (r2–)	rule (r3–)	rule (r1–)	rule (r2–)
$\alpha \vdash - \alpha \vdash \emptyset$	$\emptyset \vdash \alpha \vdash - \alpha$	$- \alpha \vdash \alpha \vdash \emptyset$	$\emptyset \vdash - \alpha \vdash \alpha$
rule (r2–)	rule (r1–)	rule (r3–)	rule (r2–)
$\alpha \vdash - - \alpha \vdash \emptyset$	$- - \alpha \vdash \alpha \vdash \emptyset$	$\emptyset \vdash \alpha \vdash - - \alpha$	$\emptyset \vdash - -$ $\alpha \vdash \alpha$

rule (r2 ≡)

$$\emptyset \vdash \alpha \equiv - - \alpha \vdash \emptyset$$

Thus the formula $\alpha \equiv - - \alpha$ is a theorem.

2. Sobociński's n–sequential propositional calculus.

In [18] Sobociński constructed and axiomatized his n–valued propositional logic with $n-1$ designated values, and negation $\sim$ and implication $\Rightarrow$ as primitive connectives.

The schemes of the rules for introducing the connectives $\sim$ and $\Rightarrow$ into the sequents Γ_k of the sequent $\Sigma = \Gamma_1 \vdash \ldots \vdash \Gamma_n$ are as follows:

for $f = 1$

$$(r1 \sim) \quad \frac{\Gamma_1 \vdash \Gamma_2 \vdash \ldots \vdash \Gamma_{n-1} \vdash \Gamma_n, \alpha}{\Gamma_1, \sim \alpha \vdash \Gamma_2 \vdash \ldots \vdash \Gamma_{n-1} \vdash \Gamma_n},$$

for $k = i, \quad 1 < i \leq n,$

$$(ri \sim) \quad \frac{\Gamma_1 \vdash \Gamma_2 \vdash \ldots \vdash \Gamma_{i-1} \vdash \Gamma_i, \alpha \vdash \Gamma_{i+1} \vdash \ldots \vdash \Gamma_n}{\Gamma_1 \vdash \Gamma_2 \vdash \ldots \vdash \Gamma_{i-1} \vdash \Gamma_i \vdash \Gamma_{i+1}, \sim \alpha \vdash \ldots \vdash \Gamma_n},$$

Now let α and β be arbitrary formulae.
For $\Sigma = \Gamma_1 \vdash \Gamma_2 \vdash \ldots \vdash \Gamma_n$, we define a sequent $\Sigma_{jk} = \Gamma'_1 \vdash \Gamma'_2 \vdash \ldots \vdash \Gamma'_n$ by putting

$$\Gamma'_i = \begin{cases} \Gamma_i & \text{if } i \neq j, i \neq k \text{ and } j \neq k \\ \Gamma_i, \alpha & \text{if } i = j \text{ and } j \neq k \\ \Gamma_i, \beta & \text{if } i = k \text{ and } j \neq k \\ \Gamma_i, \alpha, \beta & \text{if } i = j = k \end{cases}$$

Then, for $1 \leq k < n$,

$$(rk \Rightarrow)\ \frac{\{\Sigma_{jk} : 1 \leq j \leq n \text{ and } j \neq k\}}{\Gamma_1 \vdash \Gamma_2 \vdash \ldots \vdash \Gamma_k, \alpha \Rightarrow \beta \vdash \ldots \vdash \Gamma_n},$$

and

$$(rn \Rightarrow)\ \frac{\{\Sigma_{jk} : j = k \vee k = n\}}{\Gamma_1 \vdash \Gamma_2 \vdash \ldots \vdash \Gamma_k \vdash \ldots \vdash \Gamma_n, \alpha \Rightarrow \beta},$$

A propositional formula α is a theorem in Sobocinski's n–sequential propositional calculus if and only if there exists a set of overfilled n–sequents, from which the following terminal n–sequent is provable:

$$\alpha \vdash \emptyset \vdash \emptyset \vdash \ldots \vdash \emptyset$$

In general, the completeness theorem for the n–sequential propositional logic holds:

Theorem 1.

A formula α is a theorem of the n–sequential propositional calculus if and only if α is a tautologiy.

Proof.

Let α be first unprovable, and assume it is so because in each proof tree with a terminal sequent

$$\alpha \vdash \emptyset \vdash \emptyset \vdash \ldots \vdash \emptyset$$

there is a non–overfilled initial sequent; the other case can be handled similarly. Now let T be a maximal proof tree with $\alpha \vdash \emptyset \vdash \emptyset \vdash \ldots \vdash \emptyset$ as a terminal sequent. The maximality implies that the initial sequents in T do not contain logical connectives, and the assumption implies that there exists a maximal branch B of T consisting of non–overfilled sequents only. Let us write down the sequents of B in descending order:

$$\Sigma^1, \Sigma^2, \ldots, \Sigma^m = \alpha \vdash \emptyset \vdash \emptyset \vdash \ldots \vdash \emptyset$$

Then, for each $1 \leq k \leq m, \Sigma^k$ appears as one of the premises in some rule schema with conclusion Σ^{k+1}.
Now let us define a valuation $v : V \to \mathbf{E}$ as follows.
For propositional variables not appearing in $\Sigma^1 = X_1^1 \vdash \ldots \vdash X_n^1$ we fix their values arbitrarily; otherwise $v(p) = i$ if and only if $p \in X_1^1$. Since Σ^1 is not overfilled, this defines v unambiguously. Moreover, Σ^1 contains no logical connectives, therefore the following claim holds for $k = 1$:

Claim. Let $\Sigma^k = X_1^k \vdash X_2^k \vdash \ldots \vdash X_n^k$. For each $i \in \mathbf{E}$, if $\beta \in X_i^k$, then $\|\beta\|(v) = i$

Next we check the Claim for each $k \leq m$. Suppose it holds for some $k < m$. Since Σ^{k+1} is the conclusion of a rule schema, and Σ^k is one of its premises, $\beta \in X_i^{k+1}$ implies $\beta \in X_i^k$ except for exactly one formula β.

Assume $\beta \in X_i^{k+1}$. Now, if $\beta = \omega(\alpha_1, \ldots, \alpha_r)$, then by the definition of the rule schemas $\alpha_1 \in X_{jl}^k, \ldots, \alpha_r \in X_{jr}^k$ for some $j_1, j_2, \ldots, j_r \in \mathbf{E}$ such that

$$(op_r^S(\omega))(j_1, \ldots, j_r) = i$$

Hence the induction hypothesis yields,

$$\|\alpha_1\|(v) = j_1, \ldots, \|\alpha_r\|(v) = j_r,$$

and thus

$$\|\beta\|(v) = (op_r^S(\omega))(\|\alpha_1\|\alpha_r\|(v)),$$

proving the Claim for $k+1$. Since $\{\alpha\} = X_1^m$, for $k = m$ the Claim gives $\|(v)\| = 1$, which shows that α is not a tautology. Therefore, if α is a tautology, then it is a theorem of the n–sequential propositional calculus.

To prove the opposite implication, suppose that α is provable, but it is not a tautology. Then there exists a valuation $v : V \to \mathbf{E}$ such that $\|\alpha\|(v) < 1$ for $2 \leq 1 \leq n$. It can be checked as above that if the Claim holds for some sequent in the derivation tree, then it holds for at least one of its predecessor sequents. Consequently, there is a maximal branch in the proof tree on which the Claim holds. Thus there is a branch in the proof tree on which the Claim holds. Hence this branch cannot contain an overfilled sequent, which is a contradiction.

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Multi-valued n–sequental propositional logic

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Abstract

The aim of this paper is the presentation of any finite – valued propositional logic by means of the so – called n–sequential method, where an n–sequent is defined as an ordered n–tuple of finite set of formula. This method allows us to elaborate a program for automatic theorem proving within the chosen logic. Such a program was worked out by the author and T. Kudła at the Institute of Mathematics of the University of Częstochowa.