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On one Linear Approximation Used in one Applied Problem

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Abstract:
The purpose of this talk is to demonstrate how useful is the linear approximation

$$\left(1-x^2\right)^{-\frac{1}{2}} \doteq 1+\frac{1}{2}x^2$$

for the easy solution of the problem of conversion of mass to energy.

The importance of approximations stems from the fact that so many of the functions we deal with in science and engineering are too complicate. On the other hand, in these cases we can sometimes approximate such functions with simpler ones that gives the accuracy enough to solve problems described by complicated functions. The aim of this article is to show the simplest of the useful approximations – the standard linear approximation enables us to understand the relation between the mass and the energy.

Let us start with Einstein's formula for the mass of a body

$$m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}},$$

where m_0 represents the mass of this body if it is not moving, v denotes the velocity of the body and c is the speed of the light. We can do with the above formula whatever, i.e. we could ask for the increment $\Delta m = m - m_0$, i.e. for the increase in mass that results from the added velocity v.

Thus,

$$\Delta m = m - m_0 = m_0\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right) = m_0\frac{1-\sqrt{1-\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Probably we can continue with a routine arithmetic but nothing reasonable will be seen – neither as for the mathematical expression for the increment Δm nor as for the physical interpretation of the increase in mass.

Let us leave our problem concerning of Δm and remember our knowledge of the linearization and (standard) linear approximation:

If the function $y = f(x)$ is differentiable at $x = a$, then

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** *of f at a.*

The approximation

$$f(x) \doteq L(x)$$

is the **(standard) linear approximation** *of f at a.*

There is no problem, assuming the Power Rule

$$\frac{d}{dx}(1 + x)^n = n(1 + x)^{n-1}$$

valid for any real number and $1 + x > 0$, to show that the linearization of the positive function $f(x) = (1 + x)^n$ at the point $a = 0$ is $L(x) = 1 + nx$, i.e.

$$(1 + x)^n \doteq 1 + nx.$$

This linear approximation is good for values od x near zero and is frequently used especially in physics and engineering. We will show how useful is this approximation for our problem with the increment of the mass of a body Δm.

First, if we substitute $-x^2$ for x and $-\frac{1}{2}$ for n in the linear approximation

$$(1 + x)^n \doteq 1 + nx,$$

we get

$$\left(1 + \left(-x^2\right)\right)^{\frac{1}{2}} \doteq 1 + \left(-\frac{1}{2}\right)\left(-x^2\right) = 1 + \frac{1}{2}x^2.$$

If we treat $(1 + (-x^2))^{-\frac{1}{2}}$ as quotient, we have

$$\frac{1}{\sqrt{1 - x^2}} \doteq 1 + \frac{1}{2}x^2$$

Second, let's go back to the formula for Δm

$$\Delta m = m - m_0 = m_0 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right).$$

When v is very small compared to c (as common in real life situations), $\frac{v^2}{c^2}$ is close to zero and it is possible to use the above standard linear approximation

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \doteq 1 + \frac{1}{2}\frac{v^2}{c^2}$$

(we substituted $\frac{v}{c}$ for x) to write

$$\Delta m \doteq m_0 \frac{1}{2} \frac{v^2}{c^2},$$

or

$$(\Delta m)c^2 \doteq \frac{1}{2} m_0 v^2.$$

Third, in Newtonian physics, $\frac{1}{2} m_0 v^2$ is the kinetic energy (K) of the body, and if we rewrite the last expression in the form

$$(\Delta m)c^2 \doteq \frac{1}{2} m_0 v^2 - 0 = \frac{1}{2} m_0 v^2 - \frac{1}{2} m_0 0^2,$$

we see that

$$(\Delta m)c^2 \doteq \Delta K.$$

In other words, the change in kinetic energy ΔK in going from velocity 0 to velocity v is equal $(\Delta m)c^2$, i.e. ΔK equals the increment of the mass of a body times the square of the speed of light.

Conclusion: By means of the linear approximation we have found what the conversion of mass to energy looks like. Amazing, if we remember we had started to „play" with Einstein's formula $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$.

Reference

[1] G. B. Thomas, JR., R. L. Finney, *Calculus and Analytic Geometry*, Addison – Wesley Publishing Company, Reading, Massachusetts.

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