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Remarks on BCI-algebras

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Introduction. A general algebra $(G,*,0)$ of type $(2,0)$ is called *a BCI-algebra* if the following conditions are satisfied:

(1) $((x*y)*(x*z))*(z*y)=0$,
(2) $(x*(x*y))*y=0$,
(3) $x*x=0$,
(4) $x*y=y*x=0$ *implies* $x=y$,
(5) $x*0=0$ *implies* $x=0$.

If an algebra $(G,*,0)$ of type $(2,0)$ satisfies conditions (1), (2), (3), (4) and

(6) $0*x=0$,

then it is called *a BCK-algebra.*

Every BCK-algebra is a BCI-algebra (see [6]), but *there are BCI-algebras which are not BCK-algebras.* BCI-algebras which are not BCK-algebras are called *proper.* For example, every abelian group $(G,+,0)$ defines on the set G a proper BCI-algebra with the operation $x*y=x-y$. Such representation have all BCI-algebras which are quasigroups (cf. [3]), i.e. BCI-algebras in which for every a there exists only one x such that $ax=0$, or equivalently, BCI-algebras satisfying $x*(x*y)=y$ (cf. [2]). The class of such BCI-algebras forms a variety (cf. [3]).

It is easy to verify that *a group of the exponent* 2 *is a proper BCI-algebra.* On the other hand, every BCI-algebra satisfying the condtion $0*x=x$ or the condition $(x*y)*z=x*(y*z)$ is a group in which $x*x=0$ (see [5]). Also every *para-associative BCI-algebra*, i.e. a BCI-algebra satisfying *the* (i,j,k)*-associative law*

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$$(7)\quad (x_1 * x_2) * x_3 = x_i * (x_j * x_k),$$

where $\{i, j, k\}$ is a fixed permutation of $\{1, 2, 3\}$, is a group of the exponent 2 (see [1]).

1. Alternative and flexible BCI-algebras

In this part we shall investigate BCI-algebras in which one of the following conditions is satisfied:

$$(8)\quad (y * x) * x = y * (x * x),$$
$$(9)\quad (x * x) * y = x * (x * y),$$
$$(10)\quad (x * y) * x = x * (y * x)$$

(i.e. *right alternative, left alternative* and *flexible BCI-algebras*).

Lemma 1. *A right alternative BCI-algebra* $(G, *, 0)$ *is a Boolean group* .

Proof. Since elements are arbitrary, then (8) implies $(y * x) * x = y * 0$, $(0 * x) * x = 0$ and $0 * x = x * 0$. Putting $y = 0$ in (2) we obtain $(x * (x * 0)) * 0 = 0$, which implies $x * (x * 0) = 0$ (by (5)). Hence $0 = x * (x * 0) = x * (0 * x)$ and $(0 * x) * x = 0$ imply $x = 0 * x$ (by (4)). Theorem 2 from [5] completes the proof. □

Lemma 2. *A left alternative BCI-algebra* $(G, *, 0)$ *is a Boolean group.*

Proof. If a BCI-algebra $(G, *, 0)$ satisfies (9), then $0 = x * (x * 0)$ and $0 * x = x * 0$. From (2), (9) and (3) follows

$$0 = (x * (x * y)) * y = ((x * x) * y) * y = (0 * y) * y\ .$$

Therefore

$$0 = x * (x * 0) = (0 * x) * x = (x * 0) * x\ .$$

Hence $x = x * 0 = 0 * x$ (by (4)), which implies (cf. [5]) that $(G, *, 0)$ is a group of the exponent 2. □

Lemma 3. *A flexible BCI-algebra* $(G, *, 0)$ *is a Boolean group.*

Proof. Because $(G, *, 0)$ satisfies (10), then $0 * x = x * 0$. Using (2), (3) and (10) we obtain

$$0 = (x * (x * x)) * x = (x * 0) * x = x * (0 * x) = x * (x * 0),$$

which gives $0 = (x*0)*x = x*(x*0)$. This by (4) implies $x = x*0 = 0*x$. Theorem 2 from [5] completes the proof. □

Since every group of the exponent 2 is an associative BCI-algebra in which (8), (9) and (10) are satisfied, then from the above lemmas follows

Theorem 1. *A BCI-algebra* $(G, *, 0)$ *is right alternative, left alternative or flexible if and only if it is a group of the exponent* 2. □

2. Łukasiewicz algebras

Any BCK-algebra $(G, *, 0)$ can be considered as a partially ordered groupoid (cf. for example [7]). This partial order is defined by the formula:

$$x \leq y \Longleftrightarrow x * y = 0.$$

Obviously, a BCI-algebra can be partially ordered by the same order, but 0 in a BCI-algebra is not the smmallest element, in general. It is only the minimal element.

A BCK-algebra $(G, *, 0)$ is called *commutative* if $x \wedge y = y \wedge x$ for all $x, y \in G$, where $x \wedge y$ is defined as $y * (y * x)$. If in a BCK-algebra $(G, *, 0)$ for all $x, y \in G$ there exists $z \in G$ such that $x \leq z$ and $y \leq z$, then this BCK-algebra is called *directed.*

Every commutative BCK-algebra is a lower semilattice with respect to $\wedge$ (cf [7]). *A commutative directed BCK-algebra is a distributive lattice with respect to* $\wedge$ *and* $\vee$, where $x \vee y$ is defined as $c * ((c * x) * (c * y))$ and c is any upper bound for x and y (cf. [9]). Moreover (cf. [8]), *a commutative directed BCK-algebra is a Łukasiewicz algebra,* i.e. a general algebra $(G, *, 0)$ of type $(2, 0)$ such that

(11) $((x * z) * (x * y)) * (y * z) = 0$,
(12) $x * 0 = x$,
(13) $(x * y) * x = 0$,
(14) $x * (x * y) = y * (y * x)$,
(15) $x * y = (x * y) * (y * x)$

hold for all $x, y, z \in G$.

The converse statement is not true. *There are Lukasiewicz algebras which are not directed BCK-algebras.* For example, the set $G = \{0, 1, 2\}$ with the operation defined as follows

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

is a Łukasiewicz algebra. It is also a commutative BCK-algebra, but it is not directed because 1 and 2 have not an upper bound.

Theorem 2. *Every Łukasiewicz algebra is a commutative BCK-algebra.*

Proof. The condition (11) is the same as (1). Putting $z = 0$ in (11) we obtain $((x*0)*(x*y))*(y*0) = 0$, which (by (12)) implies $(x*(x*y))*y = 0$, i.e. (2) is satisfied. Putting $x = y = 0$ in (11) and using (12) we obtain (3). Now, if $x * y = y * x = 0$, then (12) and (14) imply

$$x = x * 0 = x * (x * y) = y * (y * x) = y * 0 = y .$$

Hence (4) holds. Replacing y in (14) by 0 and using (12) with (13) we obtain $0 = 0 * (0 * x)$. This (by (2)) implies $0 = (0 * (0 * x)) * x = 0 * x$, which proves (6). The commutativity follows from (14). □

Corollary 1. *A Łukasiewicz algebra is a BCI-algebra.* □

Corollary 2. *A para-associative Łukasiewicz algebra is trivial, i.e. has only one element.*

Proof. Because a Łukasiewicz algebra $(G, *, 0)$ is a BCI-algebra, then the para-associativity implies that this algebra is a group in which $x * x = 0$ (cf. [1]). Thus

$$0 = (x * y) * x = (y * x) * x = y * (x * x) = y * 0 = y$$

by (13) and (12), which completes the proof. □

As a simple consequence of Theorem 1 we obtain

Corollary 3. *A right alternative (respectively: left alternative or flexible) Łukasiewicz algebra is trivial.* □

Observe that the axioms system (11) – (15) for a Łukasiewicz algebra is not independent. Indeed, in every BCI-algebra we have $(x*y)*z = (x*z)*y$

(cf. [7]), which implies $(x*y)*x=(x*x)*y=0*y=0$. Hence a Łukasiewicz algebra can be defined by (11), (12), (14) and (15), i.e. (13) may be omitted.

3. Weak BCC-algebras

By *a weak BCC-algebra* we mean an algebra $(G,*,0)$ of type (2) satisfying (3), (4),

(16) $((x*y)*(z*y))*(x*z)=0$,
(17) $x*0=x$.

Every BCI-algebra is a weak BCC-algebra, but not conversely (cf [10]). One can prove (cf. [10]) that a weak BCC-algebra is a BCI-algebra iff it satisfies (2). A weak BCC-algebra is called *a BCC-algebra* if it also satisfies (6). A BCC-algebra satisfying (2) is obviously a BCK-algebra.

Proposition 1. *An associative weak BCC-algebra is a Boolean group.*

Proof. If a weak BCC-algebra is associative, then (18) may be written in the form $((x*y)*z)*((y*x)*z)=0$, which for $z=0$ implies $(x*y)*(y*x)=0$. This, by symmetry and (4), gives $x*y=y*x$. Thus an associative weak BCC-algebra is an abelian semigroup with the neutral element. Since for every $a,b\in G$ there exists $x=a*b\in G$ such that $a*x=b$, then this semigroup is a group. By (3) it is a Boolean group. □

Proposition 2. *A right alternative weak BCC-algebra is a Boolean group.*

Proof. Putting $x=y$ in (8) we obtain $0*x=x$. Replacing in (16) x by 0, y by x and $z=y*x$ we obtain $(y*((x*y)*y))*(x*y)=0$, which together with (8) and (4) implies $x*y=y*x$. This together with (16) (for $z=0$) gives (2). Hence a weak BCC-algebra satisfying (8) is a BCI-algebra (cf. [10]). Lemma 1 ends the proof. □

Proposition 3. *A left alternative weak BCC-algebra is a Boolean group.*

Proof. Since (9) gives $0*x=x$, then (16) for $y=x$ together with (4) implies $z*x=x*z$ for all $x,z\in G$. Now, putting $x=0$ in (16) and using $x*y=y*x$ we obtain (2), which completes the proof. □

Proposition 4. *A weak BCC-algebra is a Boolean group iff it satisfies (at* *least) one of the following identities:*

(18) $x * (y * x) = y\,,$

(19) $(x * y) * x = y\,.$

Proof. A Boolean group satisfies these identities. Conversely, if a weak BCC-algebra $(G, *, 0)$ satisfies (18), then $0 * y = y$ for all $y \in G$. Moreover, the equations $b = x * a$ and $a = b * x$ have a uniquely determined solution $x \in G$. Indeed, for every $a, b \in G$ and $x = a * b$ we have $b * x = b * (a * b) = a$ and $x * a = x * (b * x) = b$. If $x * a = y * a$, then $x = a * (x * a) = a * (y * a) = y$. Thus $(G, *, 0)$ is a quasigroup. Hence (16) can be written in the form $(x * y) * (z * y) = x * z$. Therefore $(y*x)*x = (y*x)*(0*x) = y*0 = y*(x*x)$ for all $x, y \in G$, which proves that $(G, *, 0)$ is a right alternative weak BCC-algebra. By Proposition 2 it is a Boolean group.

In the case of (19) the proof is analogous. □

Theorem 3. *A para-associative weak BCC-algebra is a Boolean group.*

Proof. We shall consider six cases of the para-associativity.
1^0 The case of the $(1,2,3)$-associativity is described by Proposition 1.
2^0 Since every $(1,3,2)$-associative groupoid is also right alternative, then this case follows from our Proposition 2.
3^0 Every $(2,1,3)$-associative groupoid is left alternative. Thus such weak BCC-algebra is a Boolean group by Proposition 3.
4^0 Since the $(2,3,1)$-associativity implies $x = 0 * x$ and $x * (y * x) = y$, then this case can by reduced to Proposition 4.
5^0 The $(3,2,1)$-associativity for $x = y$ implies $0 * z = z$. For $y = 0$ it gives $x * z = z * x$. This together with (16) for $z = 0$ implies (2). Hence every $(3,2,1)$-associative weak BCC-algebra is a BCI-algebra, and in the consequence (cf. [1]), it is a Boolean group.
6^0 Analogously as in the previous case, the $(3,1,2)$-associativity implies $x * y = y * x$ and $0 * x = x$, which together with (16) gives (2). The conclusion follows from [1]. □

Proposition 5. *A commutatine weak BCC-algebra is a BCK-algebra.*

Proof. A commutative weak BCC-algebra $(G, *, 0)$ satisfies (14), which for $x = 0$ gives $0*(0*y) = 0$. This and (16) imply (6) because

$$0 * y = ((0 * 0) * (0 * y)) * (y * 0) = 0\,.$$

Moreover, $(x*y)*x = ((x*y)*(0*y))*(y*0) = 0$. But this and (14) give (2). Indeed, $(x*(x*y))*y = (y*(y*x))*y = 0$. Thus $(G,*,0)$ is a BCK-algebra. □

In the theory of BCK-algebras an important role play (cf. (cf. [7]) the following two identities:

(20) $\quad (x*y)*y = x*y\,,$

(21) $\quad x*(y*x) = x\,.$

A BCK-algebra satisfying (20) is called *positive implicative*. A BCK-algebra satisfying (21) is called *implicative*.

In the same way as Theorem 3 in [6] one can prove

Proposition 6. *A weak BCC-algebra satisfying* (20) *or* (21) *is a BCC-algebra.* □

Note that for BCC-algebras the above two identities are not equivalent. Moreover, as show the following two examples, there are positive implicative BCC-algebras which are not BCK-algebras.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	1
3	3	3	3	0

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Axioms (3), (4), (6) and (17) are obvious. Since $\{0,1,2\}$ are BCK-algebras, we must verify (16) only in the case when at least one of elements x, y, z is equal to 3. But it is a routine calculation. These algebras are not BCK-algebras because in the first we have $(2*(2*3))*3 \neq 0$. In the second $(3*(3*2))*2 \neq 0$. It is not difficult to see that these BCC-algebras are positive implicative and $x*(y*x) \neq x$ for some $x \neq y$.

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Abstract

We prove that every left or right alternative BCI-algebra is a Boolean group, but a left or right alternative Łukasiewicz algebra has only one element. Also every para-associative weak BCC-algebra is a Boolean group.

Uwagi o BCI-algebrach

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Streszczenie

Dowodzimy, że każda lewostronnie lub prawostronnie alternatywna BCI-algebra jest grupą Boole'a, ale lewostronnie lub prawostronnie alternatywna algebra Łukasiewicza ma tylko jeden element. Także każda parałączna słaba BCC-algebra jest grupą Boole'a.