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Remarks on set-theoretic relations connected with BCH-algebras

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Introduction. A general algebra $(G, *, 0)$ of type $(2,0)$ is called *a BCH-algebra* (cf. [1], [4]) if for all $x, y, z \in G$

(1) $x * x = 0$,
(2) $(x * y) * z = (x * z) * y$,
(3) $x * y = y * x = 0$ *implies* $x = y$.

Such an algebra is called *a BCK-algebra* if it satisfies the following axioms:

(4) $(x * y) * (x * z) \leq z * y$,
(5) $x * 0 = x$,
(6) $x \leq y$ *and* $y \leq x$ *imply* $x = y$,
(7) $0 \leq x$,

where $x \leq y$ is defined by $x * y = 0$. The relation $\leq$ on G is a partial order with 0 as the smallest element.

It is easily seen that *every BCK-algebra is a BCH-algebra.* An example that the converse does not hold will be given in the next section. For more information on BCH- and BCK-algebras we refer to [1], [4] and [5].

If $(G, \leq)$ is a partially ordered set with a smallest elerment 0, then G can be made into a BCK-algebra by

$$x * y = \begin{cases} 0 & if\ x \leq y, \\ x & otherwise. \end{cases}$$

This is called *the trivial structure* on $(G, \leq)$. We remark that the partial order on G considered as a BCK-algebra coincides with the original partial order.

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1. Generalities on BCH-algebras

In this section we prove elementary properties of BCH-algebras.

Lemma 1. *If* $(G, *, 0)$ *is a BCH-algebra, then* $x * 0 = 0$ *implies* $x = 0$.

Proof. Since $0 * x = (x * 0) * x = (x * x) * 0 = 0 * 0 = 0$ and $x * 0 = 0$, then (3) yields $x = 0$. □

Lemma 2. *If* $(G, *, 0)$ *is a BCH-algebra, then* $x * 0 = x$ *for every* $x \in G$.

Proof. First we remark that $(x * 0) * x = (x * x) * 0 = 0 * 0 = 0$. Now we further have:

$$\begin{array}{lll} & (x * 0) * (x * 0) = 0 & \text{by (1)} \\ \Longrightarrow & ((x * 0) * (x * 0)) * 0 = 0 & \text{by (1)} \\ \Longrightarrow & ((x * (x * 0)) * 0) * 0 = 0 & \text{by (2)} \\ \Longrightarrow & (x * (x * 0)) * 0 = 0 & \text{by Lemma 1} \\ \Longrightarrow & x * (x * 0) = 0 & \text{by Lemma 1.} \end{array}$$

As $(x * 0) * x = x * (x * 0) = 0$, then (3) yields $x * 0 = x$. □

Definition. *A BCH_0-algebra* is a BCH-algebra $(G, *, 0)$ which, for every $x \in G$ satisfies the relation $0 * x = 0$. Remark that this is actually (7) so that every BCK-algebrais a BCH_0-algebra.

Example 1. If $(G, +, 0)$ is a commutative group, then $(G, *, 0)$, where $x * y = x - y$, is a BCH-algebra. Unless $G = \{0\}$ this BCH-algebra is not a BCH_0-algebra.

Example 2. Consider the following 4-element structure given by

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	b	0

Table 1

It is easily seen that this table defines a BCH_0-algebra which is not a BCK-algebra. Indeed: $((a * c) * (a * b)) * (b * c) = a \neq 0$.

On every BCH-algebra $(G, *, 0)$ one can define a natural relation $\leq$ by $x \leq y$ if $x * y = 0$. This relation is reflexive and anti-symmetric but not transitive in general: in Table 1 we have $a \leq b$ and $b \leq c$ but *not* $(a \leq c)$. If $x \leq 0$, then $x = 0$ and if $(G, *, 0)$ is a BCH_0-algebra we have that for every $x \in G : 0 \leq x$.

In a similar way as in the BCK-algebra case we can define on every set G equipped with a distinguished point 0 and a reflexive and anti-symmetric relation ρ, a BCH-structure putting

$$x * y = \begin{cases} 0 & if\ xRy\,, \\ x & otherwise\,. \end{cases}$$

This is called *the trivial structure on* (G, ρ). This construction always yields a BCH_0-structure.

2. The main results

Proposition. *If a BCH_0-algebra $(G, *, 0)$ has a trivial structure obtained from a relation ρ and $\leq$ is the natural relation on this BCH_0-algebra, then $\rho = \leq$ only if $0\rho y$ for every $y \in G$.*

Proof. If $x \leq y$ then $x * y = 0$. This implies $x\rho y$, or $x = 0$. If the relation $0\rho y$ is satisfied, then $x \leq y$ always implies $x\rho y$, i.e. $\leq \subset \rho$. On the other hand, if $x\rho y$ then by the definition $x * y = 0$, which gives $x \leq y$. Thus, $\rho \subset \leq$ and in the consequence $\rho = \leq$. □

Example 3. We will give an example where $\rho \neq \leq$. Let $G = \{0, a\}$ and let the reflexive and anti-symmetric relation ρ by given by $0\rho 0$, $a\rho a$, *not* $(0\rho a)$ and *not* $(a\rho 0)$. Then $(G, *, 0)$ is a BCH_0algebra given by Table 2:

$*$	0	a
0	0	0
a	a	0

Table 2

The natural order of $(G, *, 0)$ satisfies $0 \leq a$. Thus $\rho \neq \leq$.

We suppose now that G is a set and ρ is a binary relation on G. We suppose moreover that there exists a distinguished point $0 \in G$ satisfying the following *minimum condition:*

$$\forall x \in G : 0\rho x\,.$$

Remark 1. If ρ is a binary relation defined on G and $0 \in G$ satisfies the above minimum condition, then we obviously have:

(a) $0\rho 0$ (*local reflexivity*)

(b) $\forall y, z \in G : 0\rho y$ *and* $y\rho z \implies 0\rho z$ (*local transitivity*).

Theorem 1. *If* $(G, *, 0)$ *, where* $*$ *is defined by the relation* ρ*, is a* BCH_0*-algebra, then* ρ *is a reflexive and anti-symmetric relation which coincides with the natural relation on the* BCH_0*-algebra* $(G, *, 0)$.

Proof. If ρ is reflexive and anti-symmetric, then the last assertion follows from Proposition. Suppose now that ρ is not reflexive, then there exists $x \in G$ such that *not* $(x\rho x)$. So $x * x = x$. However, if $(G, *, 0)$ is a BCH_0-algebra, then , $x * x = 0$, which is in contradiction with local reflexivity.

If ρ is not anti-symmetric, then there exist $x, y \in G$, $x \neq y$ such that $x\rho y$ and $y\rho x$. Then $x * y = 0$ and $y * x = 0$, which implies that $x = y$ (by (3)). This again is a contradiction. □

Theorem 2. *If* $(G, *, 0)$ *is a BCK-algebra defined by* ρ*, then* ρ *is a partial order on* G *which coincides with the partial order on the BCK-algebra* $(G, *, 0)$.

Proof. If ρ is not a partial order, then either ρ is not reflexive or ρ is not anti-symmetric or ρ is not transitive. The cases when ρ is not reflexive or ρ is not anti-symmetric lead to contradictions as shown in Theorem 1. If now ρ is not transitive, then there exist $x, y, z \in G$ such that $x\rho y$ and $y\rho z$ but *not* $(x\rho z)$. Then we have

$$x * y = 0 \,, \quad y * z = 0 \,, \quad x * z = x$$

or

$$x \leq y \,, \quad y \leq z \,, \quad x * z = x \,.$$

Transitivity of $\leq$ yields $x * z = 0$, which is in contradiction with the requirement of local transitivity. This proves the theorem. □

Remark 2. As demonstrated by the proof, local reflexivity and local transitivity suffice to obtain that ρ is a partial order. However, as shovn in Example 3, this does not suffice to conclude that ρ is the partial order obtained by looking at $(G, *, 0)$ as a BCK-algebra.

For further information concerning the relations between set-theoretic relations and structures similar to the ones studied here we refer to [2].

ADDED IN PROOF. (June 1997) This paper were written many years ago. Hence some results can be generalized in the way presented in the other our paper [3].

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Abstract

We characterize those set-theoretic relations which trivially yield the structure of BCH- or BCK-algebra.

Uwagi o teorio-mnogościowych relacjach związanych z BCH-algebrami

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Streszczenie

Charakteryzujemy te teorio-mnogościowe relacje które w prosty sposób wyznaczają strukturę BCH-algebry lub BCK-algebry.