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On some generalization of BCC-algebras

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Introduction. K.Iséki posed an interesting problem wheter the class of BCK-algebras, which are a generalization on the one hand the notion of the algebra of sets with the set subtraction as the only one fundamental non-nullary operation and on the other hand the notion of the implicational logic, is a variety. In connection with this problem Y.Komori introduced in [8] the notion of BCC-algebras and proved (using some Gentzen-type system LC) that the class of all BCC-algebras is not a variety. BCC-algebras are considered in many papers (cf. for example [1], [3] and [5]). Some generalizations of BCC-algebras are described in [1] and [4]. All these algebras are motivated by implicational logics and by propositional calculi. In many cases one can give a special translation procedure which translates obtained results into terms and well formed formulas of the corresponding logic.

Nevertheless the study of algebras motivated by known logics is interesting and very useful for corresponding logics, also in the case when the correct translation procedure not exists.

In this note by *a weak BCC-algebra* we mean a non-empty set G with a binary operation denoted by $*$ and with a constant 0 if the following conditions are satisfied:

(1) $\quad (y*z)*((x*y)*(x*z))=0\,,$

(2) $\quad x*x=0,$

(3) $\quad 0*x=x\,,$

(4) $\quad x*y=y*x=0$ *implies* $x=y\,.$

Remark that if we exchange (1) for $(x*y)*((y*z)*(x*z))=0$, we obtain the axioms system for BCI-algebras (but in the dual form).

If a weak BCC-algebra satisfes the identity

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(5) $x * 0 = 0\,,$

then it is called *a BCC-algebra.*

W.A.Dudek suggests in [4] the definition of weak BCC-algebras and BCC-algebras which is dual to the above. In his concept every BCK-algebra is a BCC-algebra; every BCI-algebra is a weak BCC-algebra but not conversely. He gives examples of weak BCC-algebras which are not BCI-algebras. But in [6] he consider BCC-algebras which coincide with our definition i.e. with the original definition given by Komori in [8].

A weak BCC-algebra is called *medial* if the condition

(6) $(x*y)*(z*u) = (x*z)*(y*u)$

holds for all $x, y, z, u \in G$. Medial BCI-algebras are described in [2], [3], [5] and [7].

Note that a medial BCC-algebra has only one element. Indeed, by (2), (3), (5) and (6), we have

$$x = 0*(0*x) = (x*x)*(0*x) = (x*0)*(x*x) = (x*0)*0 = 0$$

for all $x \in G$.

Now, we give an independent axioms system for medial weak BCC-algebras.

Proposition 1.1. *The class of all medial weak BCC-algebras forms a variety. An equational base of this variety is given by the independent axioms system:* (2), (3), (6).

Proof. Every medial weak BCC-algebra satisfies (2), (3) and (6). Conversely, if an algebra $(G\,,*\,,0)$ of type $(2,0)$ satisfies these conditions, then

$$(y*z)*((x*y)*(x*z)) = (y*z)*((x*x)*(y*z)) =$$
$$(y*z)*(0*(y*z)) = (y*z)*(y*z) = 0\,,$$

which gives (1).

To prove (4), assume that $x*y = y*x = 0$. Then

$$x = 0*(0*x) = (y*x)*(0*x) = (y*0)*(x*x) =$$
$$(y*0)*0 = (y*0)*(y*y) = (y*y)*(0*y) = 0*(0*y) = y\,.$$

Hence $x = y$. Therefore $(G\,,*\,,0)$ is a medial weak BCC-algebra.

Now, we consider algebras given by Table 1, 2 and 3.

*	0	a
0	0	a
a	a	a

Table 1

*	0	a
0	0	0
a	0	0

Table 2

*	0	a
0	0	a
a	0	0

Table 3

The algebra defined by Table 1 satisfies (3) and (6), but $a * a = a$. Therefore (2) is independent. If the algebra is given by Table 2, then (2) and (6) hold, but $0 * a \neq a$. Hence (3) is independent. It is easily verify that the algebra defined by Table 3 is a BCC-algebra, but it is not medial, which proves that (6) is independent. Therefore the proof of Proposition 1.1 is complete. □

From the above proposition we obtain:

Corollary 1.2. *In any medial weak BCC-algebra the following identities hold:*

$$(7) \quad x * (y * z) = y * (x * z),$$
$$(8) \quad x * y = (y * x) * 0,$$
$$(9) \quad (y * 0) * 0 = y,$$
$$(10) \quad (x * y) * y = x.$$

Proof. By (3) and (6) we have

$$x * (y * z) = (0 * x) * (y * z) = (0 * y) * (x * z) = y * (x * z).$$

Hence (7) is satisfied. By (2), (3) and (6) we get

$$x * y = 0 * (x * y) = (y * y) * (x * y) = (y * x) * (y * y) = (y * x) * 0,$$

which proves (8).

Putting $x = 0$ in (8), we obtain (9). From (9) follows

$$(x * y) * y = (x * y) * (0 * y) = (x * 0) * (y * y) = (x * 0) * 0 = x,$$

which gives (10). □

As an immediate consequence of (8), we obtain:

Corollary 1.3. *Let G be a medial weak BCC-algebra. If A is a subalgebra of G, then $x * y \in A$ if and only if $y * x \in A$.* □

Now, we give two simple examples of medial weak BCC-algebras.

Examples

1. Every Boolean group is a medial weak BCC-algebra.

2. The algebra defined by next Table is a medial weak BCC-algebra.

$*$	0	a	b
0	0	a	b
a	b	0	a
b	a	b	0

1. Direct products and quotient algebras

A non-empty subset A of a BCC-algebra G is called *an ideal*, if (1) $0 \in A$ and (2) $x\,,\, y * x \in A$ imply $y \in A$.

Lemma 2.1. *Every subalgebra of a medial weak BCC-algebra is an ideal.*

Proof. Let A be a subalgebra of G. If $x \in A$ and $y * x \in A$, then $x * y \in A$ (Corollary 1.3) and $y = ((y * x) * x) \in A$ by (10). Hence A is an ideal. □

If A and B are subalgebras of a weak BCC-algebra G, then G is called *the direct product of A and B* if $G = A * B$ and $A \cap B = \{0\}$.

Proposition 2.2. *Let A and B be subalgebras of a medial weak BCC-algebra G. Then G is a direct product of A and B if and only if each element $x \in G$ can by uniquely expressed in the form $x = a * b$, where $a \in A$ and $b \in B$.*

Proof. If G is the direct product of A and B, then $G = A * B$. Therefore for any $x \in G$ there exists $a \in A$ and $b \in B$ such that $x = a * b$.

If $x = a * b = c * d$ for some $a, c \in A$ and $b, d \in B$, then

$$(a * c) * (b * d) = (a * b) * (c * d) = x * x = 0 \in A \cap B \subset A\,.$$

Since A is a subalgebra and an ideal, then $a * c$, $c * a$, $b * d, d * b \in A$. In the same manner, we prove that these elements belong to B. Hence $a * c$, $c * a$, $b * d$, $d * b \in A \cap B = \{0\}$. Therefore $a * c = c * a = 0$ and $b * d = d * b = 0$, which gives $a = c$ and $b = d$. Hence this representation is uniquely determined by x.

On the other hand, if each $x \in G$ has a unique representation in the form $x = a * b$, where $a \in A$ and $b \in B$, then $G = A * B$.

To prove $A \cap B = \{0\}$, observe that if $x \in A \cap B$, then $(x*0)*0 = x$ (by (9)) and $0*x = x$. The uniqueness of this representation implies $x = 0$. Hence $A \cap B = \{0\}$. □

Proposition 2.3. *Let G be a medial waek BCC-algebra. The relation $\sim$ is a congruence on G if and only if there exists a subalgebra A of G such that $x \sim y \Longleftrightarrow x*y \in A$.*

Proof. Let A be a subalgebra of G and let $\sim$ be the relation defined on G as follows: $x \sim y \Longleftrightarrow x*y \in A$. Then $\sim$ is a congruence on G. Indeed, since $0 \in A$, then $x \sim x$. Hence this relation is reflexive. By Corollary 1.3 it is also symmetric. We prove the transitivity. If $x \sim y$ and $y \sim z$, then $x*y,\ y*z \in A$. Since A is a subalgebra, then by (7) and (10) we have $x*z = x*((z*y)*y) = (z*y)*(x*y) \in A$. Hence $x \sim z$, which proves the transitivity. The substitution property follows from (6).

Conversely, let $\sim$ be a congruence on G and let $A = \{x \in G : 0 \sim x\}$. Of course $0 \in A$. If $x, y \in A$, then $0 \sim a$ and $0 \sim y$, which implies $0 \sim x*y$. Hence $x*y \in A$, which shows that A is a subalgebra of G.

Now, we prove that $x \sim y \Longleftrightarrow x*y \in A$. If $x \sim y$, then $0 = x*x \sim x*y$ because $\sim$ is reflexive and transitive. Therefore $0 \sim x*y$, i.e. $x*y \in A$. On the other hand, if $x*y \in A$, then $0 \sim x*y$. Since $y*y$, then $0*y \sim (x*y)*y$, which implies $y \sim x$. This completes the proof. □

From the above result follows that G may be decomposed by this equivalence relation into disjoint classes. The class containing x will be denoted by A_x. Let $x*A = \{x*a : a \in A\}$ be the left coset of A in G and let $A*x = \{a*x : a \in A\}$ be the right coset.

Proposition 2.4. *If the congruence $\sim$ is defined by a subalgebra A, then*

(i) $A_a = A_0 = A$ *for all* $a \in A$,

(ii) $A_x = A*x$,

(iii) $A*x = (x*0)*A$,

(iv) $(A*x)*0 = x*A$,

(v) $A*x = A*y$ *if and only if* $x*y \in A$,

(vi) $x*A = A*(x*0)$,

(vii) $x*A = A_y$ *for* $y = x*0$.

Proof. *(i)* Since $\sim$ is an equivalence relation, then $A_x = A_y$ or $A_x \cap A_y = \emptyset$. If $a \in A$, then $0 \sim a$. Thus $a \in A_a \cap A_0$ and $A_a = A_0$. The equation $A_0 = A$ follows from the proof of Proposition 2.3.

(ii) If $y \in A_x$, then $y \sim x$ and in the consequence $y * x \in A$. Hence $y * x = a$ for some $a \in A$. This implies $y = (y * x) * x = a * x \in A * x$. Thus $A_x \subset A * x$.

Conversely, if $y \in A * x$, then $y = a * x$ for some $a \in A$. Therefore

$$(x * y) * (a * 0) = (x * a) * (y * 0) = y * ((x * a) * 0) = y * (a * x) = 0\,.$$

Since $a * 0 \in A$ and A is an ideal (Lemma 2.1), then $x * y \in A$, which completes the proof of this part.

(iii) If $y \in A * x$, then $y = a * x$ for some $a \in A$. Thus, by (8) and (6), we obtain

$$y = a * x = (x * a) * 0 = (x * a) * (0 * 0) = (x * 0) * (a * 0) \in (x * 0) * A\,,$$

which proves $A * x \subset (x * 0) * A$.

If $y \in (x * 0) * A$, then $y = (x * 0) * a$ for some $a \in A$. Thus, by (3), (6) and (7), we have

$$(x * y) * a = (x * y) * (0 * a) = (x * 0) * (y * a) = y * ((x * 0) * a) = 0 \in A.$$

Since A is an ideal, then $x * y \in A$, i.e. $x \sim y$. Therefore $(x * 0) * A \subset A * x$, which proves (ii).

(iv) we obtain as a simple consequence of (8).

(v) follows from (ii).

(vi) is a consequence of (9).

(vii) follows from (ii) and (vi). □

In the sequel the set $G/\sim$ of all equivalence classes $x * A$ will be denoted by G/A. Putting

$$(x * A) * (y * A) = (x * y) * A\,,$$

we define the operation $*$ in G/A. Since this formula is independent of the choise of x and y, then this operation is well-defined. It is not difficult to see that $(G/A\,, * A)$ is a weak BCC-algebra.

Corollary 2.5. *If $(G\,, *\,, 0)$ is a medial weak BCC-algebra such that G/A has only two elements, then $x * A = A * x$ for all $x \in G$.* □

Now, we prove the classical result due to Lagrange.

Proposition 2.6. *If $(G, *, 0)$ is a finite medial weak BCC-algebra, then $Card(G) = Card(A) . Card(G/A)$ for every subalgebra A.*

Proof. Let $f : A \longrightarrow A * x$ be defined as $f(a) = a * x$. If $f(a) = f(b)$, then $a * x = b * x$. Hence

$$0 = (a * x) * (b * x) = b * ((a * x) * x) = b * a$$

and

$$0 = (b * x) * (a * x) = a * ((b * x) * x) = a * b,$$

which implies (by (4)) $a = b$. Therefore f is one-to-one. Thus $Card(A) = Card(A * x)$.

Suppose now that an algebra G has n elements and a subalgebra A of G has k elements. We can decompose G into a union of a finite number of disjoint left cosets:

$$G = (x_1 * A) \cup (x_2 * A) \cup \ldots \cup (x_p * A).$$

Since each of the p cosets in the above decomposition has k elements, the set G has pk elements. Hence $n = pk$. □

2. Homomorphisms

Direct computations show that if $f : G_1 \longrightarrow G_2$ is a homomorphism of medial weak BCC-algebras G_1 and G_2, then $ker\, f = \{ x \in G_1 : f(x) = 0 \}$ is a subalgebra of G_1. A homomorphism f is one-to-one iff $ker\, f = \{0\}$. If A is a subalgebra of G_1, then the mapping $h : G_1 \longrightarrow G_1/A$ defined by $h(x) = x * A$ is a homomorphism of G_1 onto quotient weak BCC-algebra G_1/A and the kernel of h is A.

Lemma 3.1. *Let G_1 and G_2 be two medial weak BCC-algebras and let f be a homomorphism from G_1 onto G_2. If A is a subalgebra of G_1 such that $ker\, f \subset A$, then $A = f^{-1}(f(A))$.*

Proof. Let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$. Hence $f(x) = f(a)$ for some $a \in A$, which implies $f(x * a) = f(x) * f(a) = 0$. Therefore $x * a \in ker\, f \subset A$. Since a subalgebra A is an ideal, then $x \in A$. Thus $f^{-1}(f(A)) \subset A$. But the reverse inclusion always holds and so the desidered equality follows. □

Proposition 3.2. *If G_1 and G_2 are medial algebras and if f is a homomorphism from G_1 onto G_2, then there exists a one-to-one correspondence between the subalgebras A of G_1 such that $ker\, f \subset A$ and the set of all* *subalgebras B of G_2. In this case $B = f(A)$ for all subalgebras B of G_2.*

Proof. Starting with an arbitrary subalgebra B of G_2, we must produce some subalgebra A of G_1 such that $ker\, f \subset A$ and $f(A) = B$. The set $f^{-1}(B)$ is a subalgebra of G_1 and $ker\, f = f^{-1}(0) \subset f^{-1}(B)$. Since f is an onto maping, then $f(f^{-1}(B)) = B$. Hence $f^{-1}(B) = A$.

To finish of the proof, we argue that the correspondence in the question is one-to-one. Suppose then that A_1 and A_2 are subalgebras of G_1 such that $ker\, f \subset A_1$, $ker\, f \subset A_2$ and $f(A_1) = f(A_2)$. Using Lemma 3.1, we get $A_1 = f^{-1}(f(A_1)) = f^{-1}(f(A_2)) = A_2$, as desired. □

Proposition 3.3. *Let G, G_1, G_2 be medial weak BCC-algebras and let f_1 and f_2 be homomorphisms from G onto G_1 and G_2, respectively. If $ker\, f_1 \subset ker\, f_2$, then there exists a unique epimorphism $f : G_1 \longrightarrow G_2$ such that $f_2 = f \circ f_1$. Moreover, f is an isomorphism, if $ker\, f_1 = ker\, f_2$.*

Proof. For any element $f_1(x) \in G_1$ we define $f : G_1 \longrightarrow G_2$ putting $f(f_1(x)) = f_2(x)$. If $f_1(x) = f_1(y)$, then $0 = f_1(y) * f_1(x) = f_1(y * x)$. Hence $y * x \in ker\, f_1 \subset ker\, f_2$. Thus

$$f_2(x) = 0 * f_2(x) = f_2(y * x) * f_2(x) = f_2((y * x) * x) = f_2(y)\,.$$

Therefore f is a well-defined mapping. Direct computations show that f is an epimorphism and $f_2 = f \circ f_1$.

It remains to establish the uniqueness of f. Suppose that $f_2 = g \circ f_1$ for some other function $g : G_1 \longrightarrow G_2$. Then $f(f_1(x)) = f_2(x) = (g \circ f_1)(x) = g(f_1(x))$ for all $f_1(x) \in G_1$, and so $f = g$.

If $ker\, f_1 = ker\, f_2$, then f is an isomorphism. Indeed, if $f(f_1(x)) = f(f_1(y))$, then $f_2(x) = f_2(y)$ and $0 = f_2(x) * f_2(y) = f_2(x * y)$. Hence $x * y \in ker\, f_2 = ker\, f_1$. But $ker\, f_1$ is a subalgebra, then $x * y,\ y * x \in ker\, f_1$. Moreover, $0 = f_1(x * y) = f_1(x) * f_1(y)$ and $0 = f_1(y * x) = f_1(y) * f_1(x)$, which implies $f_1(x) = f_1(y)$. Thus f is a one-to-one mapping. □

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On some generalization of BCC-algebras

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Abstract

In this note we describe the class of weak BCC-algebras and prove that they form a variety. The BCC-algebras were introduced by Komori in connection with the Iséki's problem concerned with the class of BCK-algebras. We describe the class of all medial weak BCC-algebras and give a some characterization of direct products of these algebras and its quotient algebras.

O pewnym uogólnieniu BCC-algebr

Janusz Thomys

Streszczenie

W tej pracy opisujemy klasę słabych BCC-algebr i dowodzimy, że jest ona rozmaitością. BCC-algebry zostały wprowadzone przez Komoriego w związku z pewnym problemem Isékiego dotyczącym klasy BCK-algebr. Opisujemy klasę wszystkich medialnych słabych BCC-algebr i charakteryzujemy produkty proste tych algebr oraz ich algebry ilorazowe.