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A MISTAKE IN DEFINITION OF A LIMIT OF A FUNCTION AND SOME CONSEQUENCES

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Abstract

Many times our students make some errors in definitions, especially when we must apply some quantifiers. The definition of a limit is one of the definition with many quantifiers, so one can observe many mistakes in it. We want to present one of possible mistakes and show how to improve the understanding of this difficult but one of most important notions.

1. Some basic notions

We shall consider only real functions defined in an open interval.

Let us start from the classical Heine's definition of limit of a real function of a real variable.

Definition 1. *If $f\colon (a,b) \longrightarrow \mathbb{R}$ is a function and x_0 is a point from the interval (a,b), then (a real number or $-\infty$ or $+\infty$) g is called a limit of f at point x_0, if for each converging to x_0 sequence $(x_n)_{n=1}^{\infty}$ of points different from x_0, the sequence $\big(f(x_n)\big)_{n=1}^{\infty}$ is convergent to g.*

In other words one can formulate this definition as follows:

Definition 2. *If $f\colon (a,b) \longrightarrow \mathbb{R}$ is a function and x_0 is a point from the interval (a,b), then (a real number or $-\infty$ or $+\infty$) g is called a limit of f at point x_0 if for each sequence $(x_n)_{n=1}^{\infty}$ of points from (a,b) if*

(1) $\forall_{n\in\mathbb{N}}\,(x_n \neq x_0)$,
(2) $\lim\limits_{n\to\infty} x_n = x_0$,

then $\lim\limits_{n\to\infty} f(x_n) = g$.

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2. Mistake

Many times we can hear (from our students) this definition formulated in the following form:

Definition 3. *If $f\colon (a,b) \longrightarrow \mathbb{R}$ is a function and x_0 is a point from the interval (a,b), then (a real number or $-\infty$ or $+\infty$) g is a limit of f at point x_0 if there exists a converging to x_0 sequence $(x_n)_{n=1}^{\infty}$ of points different from x_0 the sequence $(f(x_n))_{n=1}^{\infty}$ is convergent to the number g.*

And symbolically:

Definition 4. *If $f\colon (a,b) \longrightarrow \mathbb{R}$ is a function and x_0 is a point from the interval (a,b), then (a real number or $-\infty$ or $+\infty$) g is called a limit of f at point x_0 if there exists a sequence $(x_n)_{n=1}^{\infty}$ of points from (a,b) such that*

(1) $\forall_{n\in\mathbb{N}}\, (x_n \neq x_0)$,
(2) $\lim\limits_{n\to\infty} x_n = x_0$,

and $\lim\limits_{n\to\infty} f(x_n) = g$.

Of course, this definition is not proper for mathematical analysis. But what can we do if after all someone formulates the definition in that form. If it happens, we can continue our lecture, for example, in this way.

3. Consequences

Let us see what happens if we take this definition into considerations. Define function f like this:

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q}, \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then numbers 0 and 1 are limits of this function at 0!(???) How is it possible. Everybody heard that there is only one limit of a function at a point!

It can happen even worse: Let ϕ be defined in the following way:

$$\phi(x) = \begin{cases} \sin\frac{\pi}{x}, & \text{if} \quad x \in \mathbb{R} \setminus \{0\}, \\ 0, & \text{if} \quad x = 0. \end{cases}$$

Now, any point from the interval $[-1,1]$ is a limit number of function ϕ.

The answer is rather strange, we are accustomed to the fact that any function has only a unique limit if it has a limit.

So we are mistaken. But never mind let's continue our lesson. Suppose, we want to know what happens if such a notion has its own name. Then let us call this notion as limit number (or limit point). In that case we take:

Definition 5. *If $f\colon (a,b) \longrightarrow \mathbb{R}$ is a function and x_0 is a point from the interval (a,b), then (a real number or $-\infty$ or $+\infty$) g is called somehow, say it is a limit number of f at point x_0 if there exists a sequence $(x_n)_{n=1}^{\infty}$ of points from (a,b) such that*

(1) $\forall_{n\in\mathbb{N}}\,(x_n \neq x_0)$,
(2) $\lim\limits_{n\to\infty} x_n = x_0$,

and

(3) $\lim\limits_{n\to\infty} f(x_n) = g$.

Let $f\colon (a,b) \longrightarrow \mathbb{R}$ be any function and x_0 be a point from (a,b). Consider any converging to x_0 sequence $(x_n)_{n=1}^{\infty}$ of points different from x_0.

There are two possibilities:
1). The sequence $\left(f\left(x_n\right)\right)_{n=1}^{\infty}$ is bounded. Then there exists a subsequence $\left(f\left(x_{k_n}\right)\right)_{n=1}^{\infty}$ which is convergent to some real number, say g. Of course, the subsequence $\left(x_{k_n}\right)_{n=1}\infty$ of the sequence $(x_n)_{n=1}^{\infty}$ is also convergent to x_0, hence g is a limit number of function f at the point x_0.
2). The sequence $\left(f\left(x_n\right)\right)_{n=1}^{\infty}$ is unbounded, for example it is unbounded from above. Then $+\infty$ is a limit number of f at the point x_0.

In this way we proved that each function has a limit number at any point of (a,b).

4. What next?

Considering Heine's method of defining limits, one can ask whether it is possible to give adequate Cauchy's condition for this notion. Without great difficulty one can get the following characterization for the case when g is a real number and x_0 is real number as well.

Theorem 1. *If $f\colon (a,b) \longrightarrow \mathbb{R}$ is a function and x_0 is a point from the interval (a,b), then a real number g is a limit number of f at point x_0 if and only if*

$$\forall_{\epsilon>0}\forall_{\delta>0}\exists_{x\in(a,b)}\Big(\big(x\in(x_0-\delta,x_0+\delta)\setminus\{x_0\}\big)\wedge f(x)\in(g-\epsilon,g+\epsilon)\Big)$$

While proving that equivalence it is time and place to remind Axiom of Choice and its importance in modern mathematics.[1]

It is important in that case, that this axiom is needed in the proof into one direction. The implication that Cauchy's condition implies Heine's condition does need assumption of Axiom of Choice. But the inverse implication makes use of that axiom.

[1] Probably, Wacław Sierpiński was the first mathematician, who had found out that the Axiom of Choice is necessary to prove the equivalence of Heine's and Cauchy's conditions of limit of a function.

5. What would happen further?

Our next considerations would follow in this way:

- Is it possible to characterize limit numbers in Cauchy's manner in the case when x_0 equals plus or minus infinity?
- Is it possible to characterize limit numbers in the case when g equals plus or minus infinity? If so:
- Characterize all other limit numbers by Cauchy's manner.
- Prove that the set of all limit numbers of any function f at any point x_0 is closed.
- Prove that a function has a limit at a point if and only if the set of all limit numbers at that point is a singleton.
- Define left sided and right sided limit numbers.
- Prove Young's theorem of asymmetry, i.e. that the set of points x at which the set of left sided limit numbers is different from the set of right-sided limit numbers is at most countable.
- Define upper and lower limits, applying the notion of limit numbers.
- Define Baire's upper and lower function of a function f.
- Prove that the set of all points of continuity of any function is G_δ-set.

Summing up, coming from an erroneous definition we were able to expand the theory of real functions and improve the understanding of a notion of a limit. So the mistake and further considerations did not waste our time. Moreover, such considerations can improve understanding of the notion of a limit.

Some of the generalizations of this notion can be found in the following articles:

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