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WHY THE AREA OF A RECTANGLE IS CALCULATED BY THE FORMULA $P = a \cdot b$?

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Abstract

It is evident for many people that the area of a rectangle can be calculated according to the very well known formula:

$$P = a \cdot b.$$

We, mathematicians do believe in no statement without the proof. Then we can ask whether it is possible to prove that this formula is correct. This article answers to that question.

1. Introduction

First of all we should discuss the problem which conditions are necessary for a function to represent area of geometric figures. Such functions are called „measures". We suppose that everybody knows what the length of a segment (of the straight line) is. We shall return to that problem soon. However, we have to define conditions for „measure" first.

With no doubt, we can say that the measure of any area should be a positive real number. Moreover, if a rectangle is divided onto two disjoint rectangles, then the area of it should be the sum of its smaller parts. Thus we can define a measure (in some kind of class of subsets of a specific space). For further information on measure theory see [1].

Definition 1. *Let $\mathcal{S}$ be a class of subsets of a fixed set X. The class $\mathcal{S}$ is called a field of sets if it fulfils the following conditions:*

(1) $\varnothing \in \mathcal{S}$,
(2) *if $A \in \mathcal{S}$ and $B \in \mathcal{S}$, then $A \cup B \in \mathcal{S}$,*
(3) *if $A \in \mathcal{S}$, then $X \setminus A \in \mathcal{S}$.*

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Definition 2. *Let $\mathcal{S}$ be a field of subsets of a fixed set X. A function $\mu\colon \mathcal{S} \longrightarrow \mathbb{R}$ is called a measure in $\mathcal{S}$ if it fulfils the following conditions:*

(4) $\mu(\varnothing) = 0$,
(5) *if $A \in \mathcal{S}$, $B \in \mathcal{S}$ and $A \cap B = \varnothing$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.*

2. A few properties of fields of sets and measure in $\mathbb{R}$

If $X = \mathbb{R}$ (or equivalently X is a straight line), then we also require that a measure should be invariant with respect to translation. For example, the intervals $[0, 1]$ and $[4, 5]$ should have the same measure. One can prove that there is no measure invariant under translation for which $\mathcal{S}$ equals to the class of all subsets of $\mathbb{R}$. Of course, there are some fields of subsets of the set of real numbers which fulfil our requirements. Even more, there exists the smallest field of subsets of $\mathbb{R}$ which contains all intervals. For such fields there exist measures fulfilling our requirements.

The same remarks are true if we replace $\mathbb{R}$ by $\mathbb{R}^2$ and intervals by rectangles.

If $A \subset \mathbb{R}$ and $q \in \mathbb{R}$, then by $A + q$ we denote the set

$$\big\{x \in \mathbb{R}\colon \exists_{a\in A}\,(x = a + q)\big\}$$

Definition 3. *If a measure μ in $\mathbb{R}$ or in $\mathbb{R}^2$ fulfils the condition*

(6) *$\mu(A + q) = \mu(A)$ for each set A from $\mathcal{S}$ and each real number q,*
(7) *$\mu([a, b]) = b - a$ for each interval $[a, b]$,*

then μ is called Jordan measure.

Theorem 1. *If $\mathcal{S}$ is a field of sets, A and B belong to $\mathcal{S}$, then $A \setminus B \in \mathcal{S}$.*

Theorem 2. *If μ is a measure (in the sense of Definition 2), $A \in \mathcal{S}$ and $B \in \mathcal{S}$, then*

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B),$$
$$\mu(A \setminus B) = \mu(A) - \mu(A \cap B).$$

3. Area of rectangles

Coming back to the main problem, let us remark, that the measure of a rectangle is a function which depends on the length of its sides. So, if we consider rectangles which sides are parallel to x and y axes, then the measure of it is a function of two variables. If we denote it by P, then this function has to fulfil the following conditions:

(8) $P(x, y) > 0$ for each positive numbers x and y,
(9) $P(x_1 + x_2, y) = P(x_1, y) + P(x_2, y)$ for each positive numbers x_1, x_2 and y,

(10) $P(x, y_1 + y_2) = P(x, y_1) + P(x, y_2)$ for each positive numbers x, y_1 and y_2.

Similar problem has been considered in [2]. We shall present the necessary calculations only.

Suppose that $P(1, 1) = a$. Of course, $a > 0$. Then the function f defined by

$$f(x) = P(x, 1)$$

fulfils the following conditions:

(11) $f(x) > 0$ for any positive number x,
(12) $f(x_1 + x_2) = f(x_1) + f(x_2)$ for every positive numbers x_1 and x_2,
(13) $f(1) = a$,
(14) if $x_1 < x_2$ then $f(x_1) < f(x_2)$.

Any function fulfilling condition (12) is called additive. Let us consider such additive function.

In view of those conditions we infer that

$$f(2) = 2 \cdot a$$

since

$$f(2) = f(1 + 1) = f1) + f(1) = a + a = 2a$$

and, by induction,

$$f(n) = n \cdot a$$

for every positive integer n.

Similarly,

$$f\left(\frac{1}{2}\right) = \frac{1}{2} \cdot a,$$

since

$$f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) = f\left(\frac{1}{2} + \frac{1}{2}\right) = f(1) = a.$$

As before, applying mathematical induction one can prove that

$$f\left(\frac{1}{n}\right) = \frac{1}{n} \cdot a$$

for every positive integer n and similarly

$$f\left(\frac{k}{n}\right) = \frac{k}{n} \cdot a$$

for every positive integers k and n.

In this way we have proved that

$$f(q) = q \cdot a$$

for each positive rational number q.

In the end, we are going to prove that

$$f(x) = x \cdot a$$

for each positive real number x. To prove this, let us assume that it is not true, which means that there exists a real (not rational) number x such that

$$f(x) \neq x \cdot a.$$

There are two possibilities:

$$f(x) < x \cdot a \quad \text{or} \quad f(x) > x \cdot a.$$

Let us consider the first case. Since $a > 0$, thus $\frac{f(x)}{a} < x$. Then there exists a rational number w such that

$$\frac{f(x)}{a} < w < x,$$

hence $f(x) < aw$, therefore $f(x) < f(w)$. Thus we get the contradiction with (14).

Similar argumentation can be done in the other case.

In such a way we have proved that $f(x) = ax$ for every positive real number x, where a is a positive constant.

Usually we assume that $a = 1$, but this time we have to consider a as a function of a real variable y i.e. the second side of a rectangle.

If we repeat the same arguments for the second variable y, we infer that the area $P(x, y)$ of the rectangle which sides have lengths x and y is given by the formula:

$$P(x, y) = x \cdot y.$$

References

[1] Halmos P. R., *Measure Theory*, Springer Verlag, New York, Heidelberg, Berlin, 1974.

[2] Sahoo P. K., Kannappan P., *Introduction to Functional Equation*, Chapman & Hall, Boca Raton, 2011.

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