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Two-dimensional and One-dimensional Balance Equations and Their Applications

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Abstract

An interfacial region and a three-phase line region are considered as two-dimensional and one-dimensional continua. Equations of the linear momentum balance and moment-of-momentum balance generalize the Laplace equation for surfaces and the Young equation for lines. Balance equations for surface dislocations and disclinations are also considered. The motor analysis is used for a description of continua with couple stresses.

1. Introduction

The classical equations of the theory of capillarity – the Laplace equation and the Young equation – represent balances of forces for a two-dimensional surface separating two bulk phases and for one-dimensional line separating three bulk and three surface phases. In addition, great attention is paid to systems in which moment effects play an important role. The first type of such systems are microemulsions [1–4]. Presence of surfactants and cosurfactants results in very low (nearly

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vanishing) surface tension [1,4,6]. In this case bending effects are of particular significance. On the other hand, bending properties are due to nonsymmetrical structure of surfactants adsorbed at an oil-water interface and an imbalance between hydrophile-water and lipophile-oil interaction [2,4,5].

The second type of such systems are lipid bilayer mambranes in which the hydrophilic polar heads are pointing toward the aqueous medium and the hydrophobic ends of the hydrocarbon chains are pointing toward the interior of the film [7,8]. The structure asymmetry of lipid bilayer due to different lipid compositions of the two constituent monolayers, the structure asymmetry of lipid monolayer itself, and the influence of different environments on two sides of the bilayer lead to considerable moment effects which demand to take couples into account.

In this paper the interfacial region and the three-phase line region are considered as two-dimensional and one-dimensional Cosserat continua with kinematics described by two independent vectors: a displacement vector $\mathbf{u}$ and a rotation vector $\boldsymbol{\omega}$. A couple-stress tensor $\boldsymbol{\mu}$ appears in such media parallel with a stess tensor $\boldsymbol{\sigma}$. Linear momentum and momemt-of-momemtum balance equations generalize the Laplace equation for surfaces and the Young equation for lines.

Experimental studies show that plastic deformation of surface layers of material begins earlier than that of the bulk [9,10]. Many authors observed drastic changes of dislocation density near the interface layer [9,10]. The dislocation velocity in surface layers exceeds that in the bulk [10–12]. Moreover, the grain boundaries and surface layers of material have their own defect structure which differs from that in a bulk [9,10,13]. Discontinuities of dislocations and surface dislocations were used by Bullough and Bilby [14] in a treatment of the theory of the crystallography of martensitic transformations. Using dislocation notions Marcinkowski [15] discussed grain boundaries, Knowles [16] studied interface boundaries, Volkov *et al.* [17,18] developed the theory of internal surfaces as autonomous elements of defect structure in crystals, Braynin *et al.* [19] studied difference dislocations in interfaces. Harris [20] considered discrete surface dislocations and disclinations and referred to liquid crystals and biological objects as possible fields of application of these ideas. Dislocation dipoles in a two-dimensional medium were used [21] to describe radiation-stimulated grain-boundary creeping. We also note the study of the process of emergence of dislocations onto the crystal surface taking into account the capillary effects [22,23].

Small thickness of the grain boundaries or surface layers allows us to model them by interfaces having their own physical and mechanical characteristics and to use the methods of continuum mechanics for description of such two-dimensional media.

The basic equations for surface dislocations and disclinations were obtained in [24,25], but in kinematic relations playing the role of balance equations for surface densities of defects the interaction of two-dimensional interface with three-dimensional phases in contact was not consitered. In this paper such an interaction is taken into account.

2. General balance equations for an interface between two phases

Consider a material volume

$$V = V_1 \cup V_2 \cup \Sigma \tag{1}$$

containing two three-dimensional phases V_1 and V_2 and two-dimensional interface Σ. The volume V has a boundary $A_1 \cup A_2 \cup L$, where the line L separates the boundary surfaces A_1 and A_2.

Any extensive quantity Ψ characterizing the volume V can be written as the sum

$$\Psi = \iiint_{V_1} \rho_1 \psi_1 \,\mathrm{d}V + \iiint_{V_2} \rho_2 \psi_2 \,\mathrm{d}V + \iint_{\Sigma} \rho_\Sigma \psi_\Sigma \,\mathrm{d}\Sigma, \tag{2}$$

where ρ_1 and ρ_2 are mass densities, ψ_1 and ψ_2 are densities of Ψ per unit mass in V_1 and V_2; ρ_Σ is the surface mass density, ψ_Σ is the density of Ψ per unit surface mass.

The time change of the extensive quantity (2) is defined by the productions Δ_1, Δ_2, Δ_Σ and the fluxes $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{J}_\Sigma$:

$$\begin{aligned}\frac{\mathrm{d}\Psi}{\mathrm{d}t} &= \iiint_{V_1} \rho_1 \Delta_1 \,\mathrm{d}V + \iiint_{V_2} \rho_2 \Delta_2 \,\mathrm{d}V + \iint_{\Sigma} \rho_\Sigma \Delta_\Sigma \,\mathrm{d}\Sigma - \\ &- \iint_{A_1} \mathbf{n}_1 \cdot \mathbf{J}_1 \,\mathrm{d}A - \iint_{A_2} \mathbf{n}_2 \cdot \mathbf{J}_2 \,\mathrm{d}A - \int_L \mathbf{N} \cdot \mathbf{J}_\Sigma \,\mathrm{d}L.\end{aligned} \tag{3}$$

Here t represents time, $\mathbf{n}_1$ and $\mathbf{n}_2$ are the outer unit normals to the boundaries A_1 and A_2; $\mathbf{N}$ is the outer unit normal to L lying in a tangent plane to Σ.

Taking into account Eq. (2) and balance equations for the volumes V_1 and V_2, differentiating the surface integral with respect to time t, using the surface divergence theorem to reduce the line integral around L to the surface integral over Σ and utilizing the two-dimensional balance equation of the surface mass density we arrive at the general two-dimensional balance equation [25,26]

$$\rho_\Sigma \frac{\mathrm{d}\psi_\Sigma}{\mathrm{d}t} = \rho_\Sigma \Delta_\Sigma - \boldsymbol{\nabla}_\Sigma \cdot \mathbf{J}_\Sigma + \mathbf{n}_1 \cdot \mathbf{J}_1 + \mathbf{n}_2 \cdot \mathbf{J}_2 +$$

$$+\rho_1(\psi_1 - \psi_\Sigma)\,(\mathbf{v}_1 - \mathbf{v}_\Sigma) \cdot \mathbf{n}_1 + \rho_2(\psi_2 - \psi_\Sigma)\,(\mathbf{v}_2 - \mathbf{v}_\Sigma) \cdot \mathbf{n}_2 \qquad (4)$$

corresponding to propagating interface and describing phase transition and wave propagation.

In Eq. (4) the surface gradient operator is used:

$$\boldsymbol{\nabla}_\Sigma = \mathbf{a}^\alpha \frac{\partial}{\partial u^\alpha}, \qquad \alpha = 1, 2, \qquad (5)$$

with u^α being surface curvilinear coordinates, $\mathbf{a}^\alpha$ being vectors of the local basis on the surface (basis vectors in the local tangential plane).

3. General balance equations for a contact line between three phases

Consider a material volume

$$V = V_1 \cup V_2 \cup V_3 \cup \Sigma_{12} \cup \Sigma_{13} \cup \Sigma_{23} \cup L \qquad (6)$$

with a boundary $A_1 \cup A_2 \cup A_3$ containing three two-dimensional interfaces Σ_{12}, Σ_{13} and Σ_{23} with boundaries L_{12}, L_{13} and L_{23} separating surfaces A_1, A_1 and A_3. This volume also contains the three phase contact line L.

Any extensive quantity Ψ characterizing the considered material volume can be written as the sum

$$\begin{aligned}\Psi = &\iiint_{V_1} \rho_1\psi_1 \,\mathrm{d}V + \iiint_{V_2} \rho_2\psi_2 \,\mathrm{d}V + \iiint_{V_3} \rho_3\psi_3 \,\mathrm{d}V + \\ &+ \iint_{\Sigma_{12}} \rho_{12}\psi_{12} \,\mathrm{d}\Sigma + \iint_{\Sigma_{13}} \rho_{13}\psi_{13} \,\mathrm{d}\Sigma + \iint_{\Sigma_{23}} \rho_{23}\psi_{23} \,\mathrm{d}\Sigma + \\ &+ \int_L \rho_L\psi_L \,\mathrm{d}L.\end{aligned} \tag{7}$$

The time change of the extensive quantity (7) is defined by the products Δ_1, Δ_2, Δ_3 in three volume phases, the products Δ_{12}, Δ_{13}, Δ_{23} in three surface phases, the product Δ_L in one line phase and the fluxes $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{J}_3$, $\mathbf{J}_{12}$, $\mathbf{J}_{13}$, $\mathbf{J}_{23}$, $\mathbf{J}_L$ in volume, surface and line phases, correspondingly:

$$\begin{aligned}\frac{\mathrm{d}\Psi}{\mathrm{d}t} = &\iiint_{V_1} \rho_1\Delta_1 \,\mathrm{d}V + \iiint_{V_2} \rho_2\Delta_2 \,\mathrm{d}V + \iiint_{V_3} \rho_3\Delta_3 \,\mathrm{d}V + \\ &+ \iint_{\Sigma_{12}} \rho_{12}\Delta_{12} \,\mathrm{d}\Sigma + \iint_{\Sigma_{13}} \rho_{13}\Delta_{13} \,\mathrm{d}\Sigma + \iint_{\Sigma_{23}} \rho_{23}\Delta_{23} \,\mathrm{d}\Sigma + \\ &+ \int_L \rho_L\Delta_L \,\mathrm{d}L - \\ &- \iint_{A_1} \mathbf{n}_1 \cdot \mathbf{J}_1 \,\mathrm{d}A - \iint_{A_2} \mathbf{n}_2 \cdot \mathbf{J}_2 \,\mathrm{d}A - \iint_{A_3} \mathbf{n}_3 \cdot \mathbf{J}_3 \,\mathrm{d}A - \\ &- \int_{L_{12}} \mathbf{N}_{12} \cdot \mathbf{J}_{12} \,\mathrm{d}L - \int_{L_{13}} \mathbf{N}_{13} \cdot \mathbf{J}_{13} \,\mathrm{d}L - \int_{L_{23}} \mathbf{N}_{23} \cdot \mathbf{J}_{23} \,\mathrm{d}L - \\ &- (\boldsymbol{\lambda} \cdot \mathbf{J}_L)_-^+ ,\end{aligned} \tag{8}$$

where $\mathbf{N}_{12}$, $\mathbf{N}_{13}$ and $\mathbf{N}_{23}$ are the unit normals to the lines L_{12}, L_{13} and L_{23} of separation of bounding surfaces A_1, A_2 and A_3 lying in the corresponding tangential planes, $\boldsymbol{\lambda}$ is the unit tangential vector to the curve L. The indices "–" and "+" denote the values of quantities at the initial and ending points of the line L (at its intersections with a boundary surface $A = A_{12} \cup A_{13} \cup A_{23} \cup L_{12} \cup L_{13} \cup L_{23}$).

Accounting for (7) and (8) and the balance equations for the volumes V_1, V_2, V_3 and interfaces Σ_{12}, Σ_{13}, Σ_{23}, differentiating the line integral with respect to time t, using the Leibniz-Newton formula and the one-dimensional equation for the line mass density we obtain the general one-dimensional balance equation [27–29]

$$\rho_L \frac{\mathrm{d}\psi_L}{\mathrm{d}t} = \rho_L \Delta_L - \boldsymbol{\nabla}_L \cdot \mathbf{J}_L + \mathbf{N}_{12} \cdot \mathbf{J}_{12} + \mathbf{N}_{13} \cdot \mathbf{J}_{13} + \mathbf{N}_{23} \cdot \mathbf{J}_{23} +$$
$$+\rho_{12}(\psi_{12} - \psi_L)\,(\mathbf{v}_{12} - \mathbf{v}_L) \cdot \mathbf{N}_{12} + \rho_{13}(\psi_{13} - \psi_L)\,(\mathbf{v}_{13} - \mathbf{v}_L) \cdot \mathbf{N}_{13} +$$
$$+\rho_{23}(\psi_{23} - \psi_L)\,(\mathbf{v}_{23} - \mathbf{v}_L) \cdot \mathbf{N}_{23}. \tag{9}$$

The line gradient operator $\boldsymbol{\nabla}_L$ is introduced by the formula

$$\boldsymbol{\nabla}_L = \boldsymbol{\lambda} \frac{\partial}{\partial s}, \tag{10}$$

where s denotes the length of a curve. The unit tangential vector $\boldsymbol{\lambda}$ as well as the principal normal $\boldsymbol{\tau}$ and binormal $\boldsymbol{\nu}$ form the Frenet trihedron.

4. The generalized Laplace and Young equations

Identifying the quantity ψ with the displacement velocity vector $\mathbf{v}$, the production Δ with the body force vector $\mathbf{f}$ and the flux $\mathbf{J}$ with the stress tensor $\boldsymbol{\sigma}$ (with the opposite sign) we obtain

$$\rho_\Sigma \frac{\mathrm{d}\mathbf{v}_\Sigma}{\mathrm{d}t} = \rho_\Sigma \mathbf{f}_\Sigma + \boldsymbol{\nabla}_\Sigma \cdot \boldsymbol{\sigma}_\Sigma - \mathbf{n}_1 \cdot \boldsymbol{\sigma}_1 + \mathbf{n}_2 \cdot \boldsymbol{\sigma}_2 +$$
$$+\rho_1(\mathbf{v}_1 - \mathbf{v}_\Sigma)\,(\mathbf{v}_1 - \mathbf{v}_\Sigma) \cdot \mathbf{n}_1 + \rho_2(\mathbf{v}_2 - \mathbf{v}_\Sigma)\,(\mathbf{v}_2 - \mathbf{v}_\Sigma) \cdot \mathbf{n}_2 \tag{11}$$

and

$$\rho_L \frac{\mathrm{d}\mathbf{v}_L}{\mathrm{d}t} = \rho_L \mathbf{f}_L + \boldsymbol{\nabla}_L \cdot \boldsymbol{\sigma}_L - \mathbf{N}_{12} \cdot \boldsymbol{\sigma}_{12} - \mathbf{N}_{13} \cdot \boldsymbol{\sigma}_{13} - \mathbf{N}_{23} \cdot \boldsymbol{\sigma}_{23} +$$
$$+\rho_{12}(\mathbf{v}_{12} - \mathbf{v}_L)\,(\mathbf{v}_{12} - \mathbf{v}_L) \cdot \mathbf{N}_{12} + \rho_{13}(\mathbf{v}_{13} - \mathbf{v}_L)\,(\mathbf{v}_{13} - \mathbf{v}_L) \cdot \mathbf{N}_{13} +$$
$$+\rho_{23}(\mathbf{v}_{23} - \mathbf{v}_L)\,(\mathbf{v}_{23} - \mathbf{v}_L) \cdot \mathbf{N}_{23}. \tag{12}$$

Identifying the quantity ψ with the rotation velocity vector $\mathbf{w}$ multiplied by the factor α not depending on time and connected with the moment of inertia, the flux $\mathbf{J}$ with the couple-stress tensor $\boldsymbol{\mu}$ (with the opposite sign) and assuming

$$\Delta_\Sigma = \rho_\Sigma \mathbf{m}_\Sigma - \left(\boldsymbol{\sigma}_\Sigma \overset{\times}{\cdot} \mathbf{a}\right)^*, \tag{13}$$

$$\Delta_L = \rho_L \mathbf{m}_L - \left(\boldsymbol{\sigma}_L \overset{\times}{\cdot} \boldsymbol{\lambda} \otimes \boldsymbol{\lambda}\right)^*, \tag{14}$$

where $\mathbf{m}_\Sigma$ is the surface couple vector, $\mathbf{m}_L$ is the line couple vector, $\otimes$ is the tensor product, $\times$ is the cross product, $\mathbf{a}$ is the first fundamental form of a surface, a dot denotes the scalar product, an asterisk marks the transpose of a tensor, we get

$$\begin{gathered}\alpha_\Sigma \rho_\Sigma \frac{\mathrm{d}\mathbf{w}_\Sigma}{\mathrm{d}t} = \rho_\Sigma \mathbf{m}_\Sigma + \boldsymbol{\nabla}_\Sigma \cdot \boldsymbol{\mu}_\Sigma - \left(\boldsymbol{\sigma}_\Sigma \overset{\times}{\cdot} \mathbf{a}\right)^* - \\ -\mathbf{n}_1 \cdot \boldsymbol{\mu}_1 - \mathbf{n}_2 \cdot \boldsymbol{\mu}_2 + \rho_1(\alpha_1 \mathbf{w}_1 - \alpha_\Sigma \mathbf{w}_\Sigma)(\mathbf{v}_1 - \mathbf{v}_\Sigma) \cdot \mathbf{n}_1 + \\ +\rho_2(\alpha_2 \mathbf{v}_2 - \alpha_\Sigma \mathbf{w}_\Sigma)(\mathbf{v}_2 - \mathbf{v}_\Sigma) \cdot \mathbf{n}_2\end{gathered} \tag{15}$$

and

$$\begin{gathered}\alpha_L \rho_L \frac{\mathrm{d}\mathbf{w}_L}{\mathrm{d}t} = \rho_L \mathbf{m}_L + \boldsymbol{\nabla}_L \cdot \boldsymbol{\mu}_L - \left(\boldsymbol{\sigma}_L \overset{\times}{\cdot} \boldsymbol{\lambda} \otimes \boldsymbol{\lambda}\right)^* - \\ -\mathbf{N}_{12} \cdot \boldsymbol{\mu}_{12} - \mathbf{N}_{13} \cdot \boldsymbol{\mu}_{13} - \mathbf{N}_{23} \cdot \boldsymbol{\mu}_{23} + \\ +\rho_{12}(\alpha_{12} \mathbf{w}_{12} - \alpha_L \mathbf{w}_L)(\mathbf{v}_{12} - \mathbf{v}_L) \cdot \mathbf{N}_{12} + \\ +\rho_{13}(\alpha_{13} \mathbf{w}_{13} - \alpha_L \mathbf{w}_L)(\mathbf{v}_{13} - \mathbf{v}_L) \cdot \mathbf{N}_{13} + \\ +\rho_{23}(\alpha_{23} \mathbf{w}_{23} - \alpha_L \mathbf{w}_L)(\mathbf{v}_{23} - \mathbf{v}_L) \cdot \mathbf{N}_{23}.\end{gathered} \tag{16}$$

We use the following order of operations in the brackets in (13) and (14):

(i) vector multiplication of the neighbouring basis vectors of the multipliers;

(ii) permutation of the second and the third basis vectors in their product:

(iii) scalar (or vector) multiplication of the first and the second basis vectors.

It should be noted that the surface stress and couple stress tensors have the following structure in the local basis $\mathbf{a}_\alpha$, $\mathbf{n}$:

$$\boldsymbol{\sigma}_\Sigma = \sigma^{\alpha\beta} \mathbf{a}_\alpha \otimes \mathbf{a}_\beta + \sigma^{\alpha n} \mathbf{a}_\alpha \otimes \mathbf{n},$$

$$\boldsymbol{\mu}_\Sigma = \mu^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta + \mu^{\alpha n}\mathbf{a}_\alpha \otimes \mathbf{n}, \tag{17}$$

while the line stress and couple stress tensors are represented in the Frenet trihedron basis as:

$$\boldsymbol{\sigma}_L = \sigma^{\lambda\lambda}\boldsymbol{\lambda} \otimes \boldsymbol{\lambda} + \sigma^{\lambda\tau}\boldsymbol{\lambda} \otimes \boldsymbol{\tau} + \sigma^{\lambda\nu}\boldsymbol{\lambda} \otimes \boldsymbol{\nu},$$

$$\boldsymbol{\mu}_L = \mu^{\lambda\lambda}\boldsymbol{\lambda} \otimes \boldsymbol{\lambda} + \mu^{\lambda\tau}\boldsymbol{\lambda} \otimes \boldsymbol{\tau} + \mu^{\lambda\nu}\boldsymbol{\lambda} \otimes \boldsymbol{\nu}. \tag{18}$$

The obtained equations generalize the classical Laplace and Young equations of the theory of capillarity taking into account the moment effects (see [27,29–31]).

Indeed, neglecting couples and assuming

$$\rho_\Sigma = 0, \quad (\mathbf{v}_1 - \mathbf{v}_\Sigma) \cdot \mathbf{n}_1 = 0, \quad (\mathbf{v}_2 - \mathbf{v}_\Sigma) \cdot \mathbf{n}_2 = 0, \quad \boldsymbol{\sigma}_\Sigma = \sigma_\Sigma \mathbf{a}_\Sigma,$$

where σ_Σ is the surface tension, $\mathbf{a}_\Sigma$ is the first fundamental form of a surface, we obtain from (11):

$$\boldsymbol{\nabla}_\Sigma \sigma_\Sigma + 2H\sigma_\Sigma \mathbf{n}_1 = \mathbf{n}_1 \cdot \boldsymbol{\sigma}_1 + \mathbf{n}_2 \cdot \boldsymbol{\sigma}_2. \tag{19}$$

Here H is the mean curvature of an interface.

Finally, if $\sigma_\Sigma =$const, then we arrive at the classical Laplace equation

$$2H\sigma_\Sigma = p_2 - p_1, \tag{20}$$

where p_1 and p_2 are the hydrostatic pressures in contacting bulk phases.

Equation (19) is a basis of theoretical investigation of various physical phenomena caused by heterogeneous surface tension including wetting of heterogeneous surfaces and interaction of surface-active melts with metals [32–37]. Extensive literature testifies to the influence of heterogeneous surface tension on various physical, mechanical and chemical processes in solids. Elwing *et al.* [38] developed a method to create a surface tension gradient along silicon or glass plates, hydrophobic at one end and hydrophilic at the other, with a gradient of wettability inbetween. In [39–41] glass beads were prepared exhibiting hydrophilic properties on one hemisphere and hydrophobic properties on the other. Raphaël [42,43] analyzed the forces which acted on the "Janus Bead" placed at the water-oil interface and discussed the behavior of a liquid strip, straddling between two different surfaces.

Cassie's study [44] has left the way open for further discussion of the wettability of heterogeneous surfaces. Such surfaces have been the object of the attention of a large number of both theoretical and experimental research workers [45–51]. Stress fields due to the surface tension gradient and their influence on wettability have been taken into account in [35,36].

When surface-active melt (indium, lead, tin, gallium and so on) interacts with metal (iron, iron-silicon alloy), a zone with a high dislocation density arises in the surface layer of the metal. For a long time it was considered [52,53] that the surface-active melt substance diffused into metal and caused the concentration stress forming a dislocation structure. However, it was shown in [54,55] that the interaction between melt and metal consists of two parts. At the later stages of this process the diffusion of melt actually occurs and the concentration stress arises, but the dislocation structure at the front of the spreading melt is determined by the early stages of interaction and the obtained results cannot be explained by the diffusion mechanism. The correlation between difference in surface tension of not-wetted and wetted parts of a metal and difference in dislocation density obtained in [56], and the observation of the fusible metal drop spreading over the iron foil in the column of an electron microscope [57] have shown that nucleation of dislocations had occurred under the wetting circumference. On these grounds one can conclude that the formation of dislocations is connected with a decrease of surface tension at the wetting perimeter [34,36,37]. The results obtained in these papers can be involved in the mechanism of refractory attack by different molten glasses [58] and the explanation of experiments on cracking of leached two-phase sodium borosilicate glass carried out in [59].

When

$$(\mathbf{v}_{12} - \mathbf{v}_L) \cdot \mathbf{N}_{12} = 0, \quad (\mathbf{v}_{13} - \mathbf{v}_L) \cdot \mathbf{N}_{13} = 0, \quad (\mathbf{v}_{23} - \mathbf{v}_L) \cdot \mathbf{N}_{23} = 0,$$

$$\boldsymbol{\sigma}_{12} = \sigma_{12}\mathbf{a}_{12}, \quad \boldsymbol{\sigma}_{13} = \sigma_{13}\mathbf{a}_{13}, \quad \boldsymbol{\sigma}_{23} = \sigma_{23}\mathbf{a}_{23},$$

$$\boldsymbol{\mu}_{12} = 0, \quad \boldsymbol{\mu}_{13} = 0, \quad \boldsymbol{\mu}_{23} = 0,$$

$$\rho_L = 0, \quad \boldsymbol{\sigma}_L = \sigma_L \boldsymbol{\lambda} \otimes \boldsymbol{\lambda},$$

we have

$$\mathbf{N}_{12}\sigma_{12} + \mathbf{N}_{13}\sigma_{13} + \mathbf{N}_{23}\sigma_{23} - \frac{\partial \sigma_L}{\partial s}\boldsymbol{\lambda} - k\sigma_L\boldsymbol{\tau} = 0, \tag{21}$$

where k is the first curvature of the contact line.

In the case of axial symmetry we get [60]

$$\sigma_{23}\cos\theta = \sigma_{13} - \sigma_{12} - \frac{\sigma_L}{R}\cos\varphi, \tag{22}$$

where R is a radius of the base of a drop, φ is the angle of inclination of the surface Σ_{23} at a contact line (the angle between the vectors $\boldsymbol{\mu}$ and $\mathbf{N}_{12}$), θ is the contact angle (the angle betweem the vectors $\mathbf{N}_{12}$ and $-\mathbf{N}_{23}$.)

If $\varphi = 0$, then we arrive at the equation

$$\sigma_{23}\cos\theta = \sigma_{13} - \sigma_{12} - \frac{\sigma_L}{R} \tag{23}$$

obtained in [61,62].

For a plane interface and $\sigma_L = 0$ the classical Young equation [63] for the contact angle θ follows from (23):

$$\sigma_{23}\cos\theta = \sigma_{13} - \sigma_{12}. \tag{24}$$

Neglecting σ_L in (21) we obtain the so-called Davidov-Neumann triangle equation [64,65]

$$\mathbf{N}_{12}\sigma_{12} + \mathbf{N}_{13}\sigma_{13} + \mathbf{N}_{23}\sigma_{23} = 0. \tag{25}$$

A large number of investigations have been devoted to the experimental study and theoretical description of the processes of wetting and spreading. Due to tensor nature of surface stresses $\boldsymbol{\sigma}_{12}$ and $\boldsymbol{\sigma}_{13}$ the shape of lying or spreading drop can deviate from the axial symmetry which was observed in many experiments. One of the first experimental papers in which such a deviation from axial symmetry was noted was [66]. The amalgams formed during spreading of a mercury drop over cadmium or zinc have elliptic shape. The deformation of tin or lead also leads to anisotropy in the spread of mercury over their surfaces. Amalgams of elliptic shape form when a drop of mercury is placed on a rolled tin foil [67,68] and the ratio of the axes depends on the degree of rolling. The eccentricity is 1.11–1.14, and the major axis of the ellipse is in the direction of the roll. The authors of the works mentioned assumed that spreading of an amalgam over the surface of a metal has the diffusion nature. Later the terms "spreading", "propagation", "diffusion" [69] and "spreading diffusion" [70] were used. However it was proved that the process under study is actually spreading [71–73]. A similar effect was detected for uniaxially deformed polymers [74,75]. When wetting the polymer by various liquids (tricresylphosphate, bromium naphthalene, ethylene glycol, formamide, glycerine) a drop of the liquid has a planar elliptic shape with the longer axis oriented in the direction of the deformation. The effect was subsequently studied in [76–78].

The anisotropy of wetting may lead to shapes other than a planar elliptic shape for the drop [79,80]. A direct connection has been established between the deviation of the surface of a semiconductor from a certain crystallographic plane and the shape of the figures of a metal spreading over the surface, making it possible to determine the crystallographic orientation of plates of a semiconductor from the spreading figures. An anisotropy of the surface being wetted can be created artificially by introducing various inhomogeneities. For example, in wetting with tin a surface consisting of ordered portions of pyroceramic of square shape on a molybdenum base the perimeter of wetting has the shape of an octahedron with rounded corners [81,82].

Using the generalized Young equation, which takes account of the tensor character of the surface tension, one can explain the anisotropy of wetting in a natural manner.

5. Continuum theory of surface dislocations and disclinations

In a three-dimensional Cosserat continuum, i.e. in a continuum which motion is described not only by the displacement vector $\mathbf{u}$ but also the independent rotation vector $\boldsymbol{\omega}$, the dislocation density tensor $\boldsymbol{\alpha}$ and the disclination density tensor $\boldsymbol{\theta}$ are defined as a departure of the plastic strain tensor $\boldsymbol{\gamma}^p$ and the plastic bend-twist tensor $\boldsymbol{\kappa}^p$ from the compatibility conditions [83]

$$\begin{aligned} \boldsymbol{\theta} &= -\boldsymbol{\nabla} \times \boldsymbol{\kappa}^p, \\ \boldsymbol{\alpha} &= -\boldsymbol{\nabla} \times \boldsymbol{\gamma}^p - \left(\boldsymbol{\kappa}^{p} {}^{\times}_{\times} \mathbf{g}\right)^*, \end{aligned} \tag{26}$$

where $\mathbf{g}$ is the metric tensor. The order of operations in the brackets in (26) is the same as described above, with the scalar multiplication substituted by the vector multiplication.

The dislocation flux tensor $\mathbf{J}$ and the disclination flux tensor $\mathbf{I}$ are introduced as [84]

$$\begin{aligned}\mathbf{I} &= \frac{\mathrm{d}\boldsymbol{\kappa}^p}{\mathrm{d}t} - \boldsymbol{\nabla}\mathbf{w}^p,\\ \mathbf{J} &= \frac{\mathrm{d}\boldsymbol{\gamma}^p}{\mathrm{d}t} - \boldsymbol{\nabla}\mathbf{v}^p - (\mathbf{w}^p \times \mathbf{g})^*.\end{aligned} \tag{27}$$

Here

$$\mathbf{v}^p = \frac{\mathrm{d}\mathbf{u}^p}{\mathrm{d}t}, \qquad \mathbf{w}^p = \frac{\mathrm{d}\boldsymbol{\omega}^p}{\mathrm{d}t}. \tag{28}$$

It follows from (26)–(28) that

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} &= -\boldsymbol{\nabla} \times \mathbf{I},\\ \frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t} &= -\boldsymbol{\nabla} \times \mathbf{J} - \left(\mathbf{I}\,{}^{\times}_{\times}\mathbf{g}\right)^*.\end{aligned} \tag{29}$$

The surface dislocation density tensor $\boldsymbol{\alpha}_\Sigma$ and the surface disclination density tensor $\boldsymbol{\theta}_\Sigma$ are defined as incompatibility of the plastic strain tensor $\boldsymbol{\gamma}^p_\Sigma$ and the plastic bend-twist tensor $\boldsymbol{\kappa}^p_\Sigma$ of the Cosserat surface [24]

$$\begin{aligned}\boldsymbol{\theta}_\Sigma &= -\boldsymbol{\nabla}_\Sigma \times \boldsymbol{\kappa}^p_\Sigma + \boldsymbol{\epsilon}_\Sigma \cdot \mathbf{b} \cdot \boldsymbol{\kappa}^p_\Sigma,\\ \boldsymbol{\alpha}_\Sigma &= -\boldsymbol{\nabla}_\Sigma \times \boldsymbol{\gamma}^p_\Sigma + \boldsymbol{\epsilon}_\Sigma \cdot \mathbf{b} \cdot \boldsymbol{\gamma}^p_\Sigma - \left(\boldsymbol{\kappa}^p_\Sigma\,{}^{\times}_{\times}\mathbf{a}\right)^*,\end{aligned} \tag{30}$$

where $\mathbf{a}$ and $\mathbf{b}$ are the first and second fundamental forms of a surface, $\boldsymbol{\epsilon}_\Sigma$ is the two-dimensional alternating tensor.

The surface dislocation flux $\mathbf{J}_\Sigma$ and the surface disclination flux $\mathbf{I}_\Sigma$ have the following form

$$\mathbf{I}_\Sigma = \frac{\mathrm{d}\boldsymbol{\kappa}^p_\Sigma}{\mathrm{d}t} - \boldsymbol{\nabla}_\Sigma \mathbf{w}^p_\Sigma, \tag{31}$$

$$\mathbf{J}_\Sigma = \frac{\mathrm{d}\boldsymbol{\gamma}^p_\Sigma}{\mathrm{d}t} - \boldsymbol{\nabla}_\Sigma \mathbf{v}^p_\Sigma - (\mathbf{w}^p_\Sigma \times \mathbf{a})^*. \tag{32}$$

The balance equations for densities of surface defects read

$$\frac{\mathrm{d}\boldsymbol{\theta}_\Sigma}{\mathrm{d}t} = -\boldsymbol{\nabla}_\Sigma \times \mathbf{I}_\Sigma + \boldsymbol{\epsilon}_\Sigma \cdot \mathbf{b} \cdot \mathbf{I}_\Sigma, \tag{33}$$

$$\frac{\mathrm{d}\boldsymbol{\alpha}_\Sigma}{\mathrm{d}t} = -\boldsymbol{\nabla}_\Sigma \times \mathbf{J}_\Sigma + \boldsymbol{\epsilon}_\Sigma \cdot \mathbf{b} \cdot \mathbf{J}_\Sigma - \left(\mathbf{I}_\Sigma \overset{\times}{\times} \mathbf{a}\right)^* . \tag{34}$$

It should be noted that the surface tensors have the following structure:

$$\begin{aligned}
\boldsymbol{\gamma}_\Sigma &= \gamma^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta + \gamma^{\alpha n}\mathbf{a}_\alpha \otimes \mathbf{n}, \qquad & \boldsymbol{\kappa}_\Sigma &= \kappa^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta + \kappa^{\alpha n}\mathbf{a}_\alpha \otimes \mathbf{n}, \\
\boldsymbol{\alpha}_\Sigma &= \alpha^{n\beta}\mathbf{n} \otimes \mathbf{a}_\beta + \alpha^{nn}\mathbf{n} \otimes \mathbf{n}, \qquad & \boldsymbol{\theta}_\Sigma &= \theta^{n\beta}\mathbf{n} \otimes \mathbf{a}_\beta + \theta^{nn}\mathbf{n} \otimes \mathbf{n}, \\
\mathbf{J}_\Sigma &= J^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta + J^{\alpha n}\mathbf{a}_\alpha \otimes \mathbf{n}, \qquad & \mathbf{I}_\Sigma &= I^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta + I^{\alpha n}\mathbf{a}_\alpha \otimes \mathbf{n}.
\end{aligned} \tag{35}$$

6. Motor calculus

In three-dimensional case the motor calculus was developed by Mises [85], the differential operators for motors we introduced in [86–89]. The motor analysis for surfaces and lines was developed by the author [90].

We use the following invariant notations for the surface gradient, divergence, and curl of surface motor $\begin{pmatrix}\mathbf{V}_\Sigma \\ \mathbf{W}_\Sigma\end{pmatrix}$ and motor-tensor $\begin{pmatrix}\mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma\end{pmatrix}$ fields:

$$\boldsymbol{\nabla}_\Sigma \begin{pmatrix}\mathbf{V}_\Sigma \\ \mathbf{W}_\Sigma\end{pmatrix} = \begin{pmatrix}\boldsymbol{\nabla}_\Sigma \mathbf{V}_\Sigma \\ \boldsymbol{\nabla}_\Sigma \mathbf{W}_\Sigma - (\mathbf{V}_\Sigma \times \mathbf{a})^*\end{pmatrix}, \tag{36}$$

$$\boldsymbol{\nabla}_\Sigma \cdot \begin{pmatrix}\mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma\end{pmatrix} = \begin{pmatrix}\boldsymbol{\nabla}_\Sigma \cdot \mathbf{Q}_\Sigma \\ \boldsymbol{\nabla}_\Sigma \cdot \mathbf{R}_\Sigma - \left(\mathbf{Q}_\Sigma \overset{\times}{\cdot} \mathbf{a}\right)^*\end{pmatrix}, \tag{37}$$

$$\boldsymbol{\nabla}_\Sigma \times \begin{pmatrix}\mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma\end{pmatrix} = \begin{pmatrix}\boldsymbol{\nabla}_\Sigma \times \mathbf{Q}_\Sigma \\ \boldsymbol{\nabla}_\Sigma \times \mathbf{R}_\Sigma + \left(\mathbf{Q}_\Sigma \overset{\times}{\times} \mathbf{a}\right)^*\end{pmatrix}. \tag{38}$$

Successive application of the surface gradient operator gives

$$\boldsymbol{\nabla}_\Sigma \times \left[\boldsymbol{\nabla}_\Sigma \begin{pmatrix}\mathbf{V}_\Sigma \\ \mathbf{W}_\Sigma\end{pmatrix}\right] = \boldsymbol{\epsilon}_\Sigma \cdot \mathbf{b} \cdot \boldsymbol{\nabla}_\Sigma \begin{pmatrix}\mathbf{V}_\Sigma \\ \mathbf{W}_\Sigma\end{pmatrix}, \tag{39}$$

$$\boldsymbol{\nabla}_\Sigma \cdot \left[\boldsymbol{\nabla}_\Sigma \times \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} \right] = \boldsymbol{\nabla}_\Sigma \cdot \left[\boldsymbol{\epsilon}_\Sigma \cdot \mathbf{b} \cdot \boldsymbol{\nabla}_\Sigma \cdot \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} \right] -$$

$$-2H\mathbf{n} \cdot \left[\boldsymbol{\nabla}_\Sigma \times \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} \right]. \tag{40}$$

The motor analogue of the surface Gauss-Ostrogradsky formula reads

$$\iint_\Sigma \boldsymbol{\nabla}_\Sigma \cdot \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} \mathrm{d}\Sigma = \int_L \mathbf{N} \cdot \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} \mathrm{d}L -$$

$$- \iint_\Sigma 2H\mathbf{n} \cdot \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} \mathrm{d}\Sigma. \tag{41}$$

We use the following invariant notations for the line gradient, divergence, and curl of line motor $\begin{pmatrix} \mathbf{V}_L \\ \mathbf{W}_L \end{pmatrix}$ and motor-tensor $\begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix}$ fields:

$$\boldsymbol{\nabla}_L \begin{pmatrix} \mathbf{V}_L \\ \mathbf{W}_L \end{pmatrix} = \begin{pmatrix} \boldsymbol{\nabla}_\Sigma \mathbf{V}_L \\ \boldsymbol{\nabla}_\Sigma \mathbf{W}_L - (\mathbf{V}_L \times \boldsymbol{\lambda} \otimes \boldsymbol{\lambda})^* \end{pmatrix}, \tag{42}$$

$$\boldsymbol{\nabla}_L \cdot \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} = \begin{pmatrix} \boldsymbol{\nabla}_L \cdot \mathbf{Q}_L \\ \boldsymbol{\nabla}_L \cdot \mathbf{R}_L - \left(\mathbf{Q}_L \overset{\times}{\cdot} \boldsymbol{\lambda} \otimes \boldsymbol{\lambda} \right)^* \end{pmatrix}, \tag{43}$$

$$\boldsymbol{\nabla}_L \times \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} = \begin{pmatrix} \boldsymbol{\nabla}_L \times \mathbf{Q}_L \\ \boldsymbol{\nabla}_L \times \mathbf{R}_L - \left(\mathbf{Q}_L \overset{\times}{\times} \boldsymbol{\lambda} \otimes \boldsymbol{\lambda} \right)^* \end{pmatrix}. \tag{44}$$

Successive application of the line gradient operator gives

$$\boldsymbol{\nabla}_L \times \left[\boldsymbol{\nabla}_L \begin{pmatrix} \mathbf{V}_L \\ \mathbf{W}_L \end{pmatrix} \right] = k\boldsymbol{\nu} \otimes \boldsymbol{\lambda} \cdot \boldsymbol{\nabla}_L \begin{pmatrix} \mathbf{V}_L \\ \mathbf{W}_L \end{pmatrix}, \tag{45}$$

$$\boldsymbol{\nabla}_L \cdot \left[\boldsymbol{\nabla}_L \times \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} \right] = -k\boldsymbol{\tau} \cdot \left[\boldsymbol{\nabla}_L \times \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} \right]. \tag{46}$$

The one-dimensional analogue of the Gauss-Ostrogradsky formula reads

$$\int_L \boldsymbol{\nabla}_L \cdot \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} \mathrm{d}L = \left[\boldsymbol{\lambda} \cdot \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} \right]_-^+ - \int_L k\boldsymbol{\tau} \cdot \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} \mathrm{d}L. \tag{47}$$

We recall that the vectors $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ form the Frenet trihedron.

Using the motor calculus the generalized Laplace and Young equations (11), (15) and (12), (16) can be written as

$$\begin{aligned}\rho_\Sigma \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{v}_\Sigma\\ \alpha_\Sigma \mathbf{w}_\Sigma\end{pmatrix} &= \rho_\Sigma \begin{pmatrix}\mathbf{f}_\Sigma\\ \mathbf{m}_\Sigma\end{pmatrix} + \boldsymbol{\nabla}_\Sigma \cdot \begin{pmatrix}\boldsymbol{\sigma}_\Sigma\\ \boldsymbol{\mu}_\Sigma\end{pmatrix} - \\ &-\mathbf{n}_1 \cdot \begin{pmatrix}\boldsymbol{\sigma}_1\\ \boldsymbol{\mu}_1\end{pmatrix} - \mathbf{n}_2 \cdot \begin{pmatrix}\boldsymbol{\sigma}_2\\ \boldsymbol{\mu}_2\end{pmatrix} + \\ &+ \begin{pmatrix}\mathbf{v}_1 - \mathbf{v}_\Sigma\\ \alpha_1 \mathbf{w}_1 - \alpha \mathbf{w}_\Sigma\end{pmatrix} \rho_1(\mathbf{v}_1 - \mathbf{v}_\Sigma)\cdot \mathbf{n}_1 + \\ &+ \begin{pmatrix}\mathbf{v}_2 - \mathbf{v}_\Sigma\\ \alpha_2 \mathbf{w}_2 - \alpha \mathbf{w}_\Sigma\end{pmatrix} \rho_2(\mathbf{v}_2 - \mathbf{v}_\Sigma)\cdot \mathbf{n}_2\end{aligned} \tag{48}$$

and

$$\begin{aligned}\rho_L \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{v}_L\\ \alpha_L \mathbf{w}_L\end{pmatrix} &= \rho_L \begin{pmatrix}\mathbf{f}_L\\ \mathbf{m}_L\end{pmatrix} + \boldsymbol{\nabla}_L \cdot \begin{pmatrix}\boldsymbol{\sigma}_L\\ \boldsymbol{\mu}_L\end{pmatrix} - \\ &-\mathbf{N}_{12} \cdot \begin{pmatrix}\boldsymbol{\sigma}_{12}\\ \boldsymbol{\mu}_{12}\end{pmatrix} - \mathbf{N}_{13} \cdot \begin{pmatrix}\boldsymbol{\sigma}_{13}\\ \boldsymbol{\mu}_{13}\end{pmatrix} - \mathbf{N}_{23} \cdot \begin{pmatrix}\boldsymbol{\sigma}_{23}\\ \boldsymbol{\mu}_{23}\end{pmatrix} + \\ &+ \begin{pmatrix}\mathbf{v}_{12} - \mathbf{v}_L\\ \alpha_{12} \mathbf{w}_{12} - \alpha \mathbf{w}_L\end{pmatrix} \rho_{12}(\mathbf{v}_{12} - \mathbf{v}_L)\cdot \mathbf{N}_{12} + \\ &+ \begin{pmatrix}\mathbf{v}_{13} - \mathbf{v}_L\\ \alpha_{13} \mathbf{w}_{13} - \alpha \mathbf{w}_L\end{pmatrix} \rho_{13}(\mathbf{v}_{13} - \mathbf{v}_L)\cdot \mathbf{N}_{13} + \\ &+ \begin{pmatrix}\mathbf{v}_{23} - \mathbf{v}_L\\ \alpha_{23} \mathbf{w}_{23} - \alpha \mathbf{w}_L\end{pmatrix} \rho_{23}(\mathbf{v}_{23} - \mathbf{v}_L)\cdot \mathbf{N}_{23}.\end{aligned} \tag{49}$$

The basic equations for the two-dimensional Cosserat surface in motor recording have the following form:

$$\begin{pmatrix}\boldsymbol{\theta}_\Sigma\\ \boldsymbol{\alpha}_\Sigma\end{pmatrix} = -\nabla_\Sigma \times \begin{pmatrix}\boldsymbol{\kappa}^p_\Sigma\\ \boldsymbol{\gamma}^p_\Sigma\end{pmatrix} + \boldsymbol{\epsilon}_\Sigma \cdot \mathbf{b} \cdot \begin{pmatrix}\boldsymbol{\kappa}^p_\Sigma\\ \boldsymbol{\gamma}^p_\Sigma\end{pmatrix}, \tag{50}$$

$$\begin{pmatrix}\mathbf{I}_\Sigma\\ \mathbf{J}_\Sigma\end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{\kappa}^p_\Sigma\\ \boldsymbol{\gamma}^p_\Sigma\end{pmatrix} - \boldsymbol{\nabla}_\Sigma \begin{pmatrix}\mathbf{w}^p_\Sigma\\ \mathbf{v}^p_\Sigma\end{pmatrix}, \tag{51}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{\theta}_\Sigma\\ \boldsymbol{\alpha}_\Sigma\end{pmatrix} = -\nabla_\Sigma \times \begin{pmatrix}\mathbf{I}_\Sigma\\ \mathbf{J}_\Sigma\end{pmatrix} + \boldsymbol{\epsilon}_\Sigma \cdot \mathbf{b} \cdot \begin{pmatrix}\mathbf{I}_\Sigma\\ \mathbf{J}_\Sigma\end{pmatrix}. \tag{52}$$

To take into account the interaction of bulk and surface phases in Eq. (52) one can introduce the following items into this equation

$$\mathbf{n}_1 \cdot \begin{pmatrix} \mathbf{I}_1 \\ \mathbf{J}_1 \end{pmatrix} + \mathbf{n}_2 \cdot \begin{pmatrix} \mathbf{I}_2 \\ \mathbf{J}_2 \end{pmatrix}$$

but due to the structure of surface defects densities and surface defects fluxes (35) we arrive at the conclusion that

$$\mathbf{n}_1 \cdot \begin{pmatrix} \mathbf{I}_1 \\ \mathbf{J}_1 \end{pmatrix} + \mathbf{n}_2 \cdot \begin{pmatrix} \mathbf{I}_2 \\ \mathbf{J}_2 \end{pmatrix} = 0. \tag{53}$$

Due to the structure of line strain tensor and bend-twist tensor dislocations and disclinations cannot exist in one-dimensional medium, but at the junction line of three surface phases the following condition

$$\mathbf{N}_{12} \cdot \begin{pmatrix} \mathbf{I}_{12} \\ \mathbf{J}_{12} \end{pmatrix} + \mathbf{N}_{13} \cdot \begin{pmatrix} \mathbf{I}_{13} \\ \mathbf{J}_{13} \end{pmatrix} + \mathbf{N}_{23} \cdot \begin{pmatrix} \mathbf{I}_{23} \\ \mathbf{J}_{23} \end{pmatrix} = 0 \tag{54}$$

is fulfilled.

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