Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XIX (2014)
129–141

MANY FACES OF MATHEMATICAL MODELLING

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Abstract

Mathematical modelling is a concept that covers a wide range of activities. Mathematical modelling can be understood both as formulation of an equation, a function, etc., describing a given situation and as a whole process of creating a model, starting from the real-world situation to the creation of a ready-to-use optimized tool. The work presents different approaches to mathematical modelling from the point of view of teaching mathematics. It presents the results of the research conducted on students (future teachers) regarding their theoretical knowledge and skills related to mathematical modelling.

1. Introduction

While studying the problem of mathematical modelling, I discovered that the concept itself is differently perceived by its users: biologists, statisticians, athletes, economists. I am interested in differences in understanding of this concept from the point of view of teaching mathematics. Depending on the level of teaching various activities can be understood as mathematical modelling. For a teacher modelling can be, for example, writing down an algebraic equation or function formulas which illustrate a given situation. On the other hand, for students (e.g. of economics), it will be a complicated process of formulating the influence of various parameters on the value of the considered variable using mathematical tools. Both interpretations comply with the definition of modelling because it covers a wide spectrum of activities which can be differently understood at different levels of education. How then define the considered notion for the purposes of teaching mathematics?

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2. The definition of mathematical modelling

W. Blum and R. Ferri define mathematical modelling as "the process of translating between the real world and mathematics in both directions" (2009, p. 45).

Z. Krygowska, who laid the foundations of Polish didactics of mathematics, defines mathematical modelling in the following way: "Mathematical modelling is an ability to describe a real situation in the language of mathematics, to interpret and verify the results in natural language, to match ready-to-use mathematical models with real situations and to search for real situations that are specific to those models, to reflect upon, analyse and evaluate one's own and the others' mathematical models. The construction of a mathematical model of the situation requires the student to examine it, and then to distinguish the objects and relations between them."

M. Niss writes concisely: "When a mathematical model is introduced (selected, modified or constructed) from scratch to deal with aspects of an extra-mathematical context and situation, we say that mathematical modelling is taking place". (2012, p. 50).

The above-mentioned statements show a wide spectrum of activities related to modelling. This term refers both to the selection of a suitable tool of solving the problem as well as the entire process of creating a new model. The quoted statements indicate the need to verify the chosen mathematical tool, control various aspects that are included in the tool and the model, verify the correctness, even by choosing sample data or considering various possible solutions to the problem, etc. Mathematical modelling is thus a process in which one cyclically returns to the initial real situation. This process is to realize which aspects of the reality have to be included in the model and which should be simplified.

Now we can see that mathematical modelling is an activity that consists of many elements. Depending on which element of this process will be given greater attention, mathematical modelling can be differently perceived.

3. Mathematical modelling in school education

W. Blum writes that "In Germany mathematical modelling is one of six compulsory competencies in the New National 'Educational Standards' for mathematics. However, in everyday mathematics teaching in most countries there is still only few modelling. Mostly 'word problems' are treated where, after 'undressing' the context, the essential aim is exercising mathematics." (Blum, Ferri, 2009, p. 47).

In Poland, mathematical modelling is one of several groups of mathematical learning objectives included in the core curriculum. In accordance with this act, students should possess the ability of mathematical modelling as they finish the fourth stage of education (age of 19). For students on mathematical education at the advanced level, this requirement is formulated as follows: a student creates the mathematical model of a given situation, taking into account limitations and reservations (Core curriculum with commentary, 2009, p. 41). However, in Polish schools, mathematical modelling is generally associated with the ability to use mathematical tools (e.g. systems of equations, linear functions) to solve the so-called word problems.

Here are some examples of tasks related to mathematical modelling indicated by the authors of the page
http://www.e-zadania.pl/matura-2012/bardzo-wazne-materialy
/standardy-wymagan-egzaminacyjnych/modelowanie-matematyczne/
which aims at preparing secondary-school graduates for their exam (matura):

1. Show number 42 as a sum of two components so that the difference of their squares equals 168.
2. Point $B = (-1, 9)$ belongs to the circle tangent to the axis Ox at the point $A = (2, 0)$. Find the equation of this circle.

In my opinion, such understanding of mathematical modelling is too narrow: tasks presented here are closed and explicitly relate to very specific mathematical issues (e.g in task 2 one has to relate to the equation of a circle known from analytic geometry). Also exercises in mathematics textbooks are generally closed due to the data; they do not require selection of information and often explicitly indicate a mathematical tool. Unfortunately, mathematical modelling is often perceived like this by teachers, which results from the fact that such understanding is imposed on them by the sources available to them.

In educational journals other faces of mathematical modelling can be found because examples of real situation appear in them. We can distinguish here two types of work:

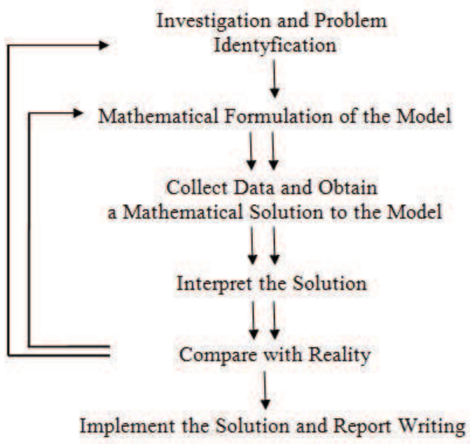
- Searching for a model based on the specific data
- Creating a model from scratch, knowing only a problem from reality

Creating a mathematical model of a phenomenon (problem) on the basis of specific data comes down to the search for functional dependencies between the data. Found model (function, equation, system of equations) allows us to predict the course of events in such moments in the future for which we may not have empirical data. The most commonly used models are functional models in the form of linear, polynomial (square in particular) and exponential functions. The search for a model can be done e.g. by a computer program which has the option of retrieving functions representing specific data. Therefore, mathematical modelling can be seen as a computer simulation performed by an appropriate program to get the formula of a function describing the considered phenomenon (see: Rybak).

Already at the stage of schooling, mathematical modelling can be understood as a search for a model from scratch, starting from the analysis of the problem, collecting sample data on which the model is created by testing and verifying the design model. Examples of school problems solved in such a way can be found in the work of P. Zarzycki (2009). Despite numerous educational benefits of unassisted creating of a mathematical model, such approach is absent from school textbooks, not to mention even the attempts to create the general model by searching for and making a symbolic notation of relations within the considered problem.

4. Mathematical modelling in university teaching

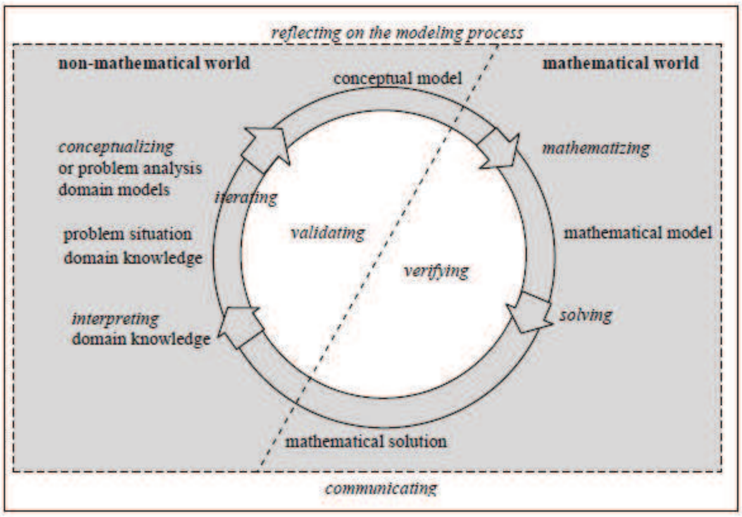
At the university level, students can meet with mathematical modelling – this applies to certain fields of study. In Poland, they are mostly technical and economic fields. Unfortunately, modelling hardly ever appears in Polish mathematical studies, even in those with teaching specializations.



Picture 1. Stages in the process of modelling (W. Morris 1967)

In the scientific literature, one can find much information both about learning and teaching mathematical modelling (Blum, Ferri, Niss, Warwick). Scientists through mathematical modelling understand the whole process of creating a model from choosing a problem (situation) to producing the final specification of a ready, verified and properly working model. In the literature one can find many diagrams showing the stages in the process of mathematical modelling. The differences in these patterns arise mostly from different levels of detail, but they always take the form of a cycle and regardless of the amount of steps shown, they describe the same process that takes into account both mathematical and non-mathematical (realistic) aspects of a given situation.

In 1967, W. Morris illustrated the process of modelling in the form of two loops consisting of 6 stages (see picture 1). The loop structure of the modelling process causes that certain stages of the process are repeated in order to create the best model.



Picture 2. The process of mathematical modelling (J. Perrent, B. Zwaneveld 2012)

In 2012, J. Perrent and B. Zwaneveld presented their diagram of mathematical modelling, which was created after a thorough analysis of a number of diagrams showing elements of mathematical modelling (see picture 2). Their draft is consistent with Morris's proposal although it was limited to a single loop. The new 'modelling cycle' clearly indicates the interpenetration of the stages (actions, activities) leading to the creation of a mathematical model. Particularly evident is the suspension of the process in two areas: in the world of mathematics and the real world. Activities performed in the first of them are known from math classes, while moving in the other requires a higher degree of skills. It results from the fact that the creator of a mathematical model should carefully analyse the problem so as to see what elements of the reality influence the analysed variable and know what dependencies there are between the analysed variables.

After mathematization and performing the calculations, a verification and validation of the results is needed. The authors of the diagram explain these activities in the following way (Perrenet, Zwaneveld, 2012, p. 18):

- Verification: The mathematical model and the solution have to be tested and adapted against mathematical logic and consistency.
- Validation: The mathematical model and the solution have to be tested and adapted against the requirements of practice.

It is these activities (among other things) that Niss (2012) thought about when he wrote that mathematical knowledge alone is insufficient for efficient creation of a mathematical model. The representations of mathematical modelling shown above cover the entire process of modelling, which should be carried out independently and not with the help of a computer program. They indicated these aspects of the process of modelling which require skills that go beyond just a deep understanding of mathematics, i.e. understanding of a real problem, creating a model that is based on reality, creating a mathematical model from a real model, interpreting mathematical results with reference to a real situation. So we can see that scientists, despite using different forms of showing what mathematical modelling is, unanimously refer to the same range of activities.

5. Mathematical modelling in the consciousness of the students of mathematics

In mathematical studies in Poland the subject of "mathematical modelling" rarely appears. This also applies to fields with teaching specialization, which is quite surprising given to the fact that as future teachers, students will be required to teach skills related to modelling, all the more so because the competence in mathematical modelling is different from pure mathematical skills, which Niss (2012) and Maaß(2006) prove in their works. So how do future teachers understand mathematical modelling if they are not quite familiar with this concept? I tried to answer this question by conducting a research.

5.1. **Methodology of the research.**

5.1.1. *The objectives of the research and the description of the research tool.* The aim of the research was to examine the knowledge and skills of students of mathematics (future teachers) related to mathematical modelling. The students were asked to solve two tasks and fill out the questionnaire. The initial 4 questions focused on the way of solving the tasks, whereas the last four focused on the students' knowledge in the field of mathematical modelling. These two types of questions were designed to examine both skills and theoretical knowledge in the field of mathematical modelling.

The research tool consisted of two cards – the first of them with the tasks to be solved and the second one with the questionnaire with questions and empty space for answers. The content of the tasks was the following:
Exc. 1 Kasia wants to prepare yoghurt sauce. For this, she went shopping. Unfortunately, on her way back home the storm started. At home, it turned out that the label on the purchased yoghurt is partially soggy and thus partially illegible. Is Kasia able to know the capacity of the yoghurt container if she read from the label the following information:

Nutritional values:	100 gr.	Container
Energy (cal.)	75	113
Fat (gr.)	1.5	2.25
Calcium (mg)	160	240

Exc. 2 Some kind of curd contains 5% of fat. Complete the following table by calculating the amount of fat in individual packs.

Pack (gr.)	Fat (gr.)
100	
150	
200	

The questions in the survey regarding the above tasks were as follows:
1. How did you solve the first task? What competence did you need to have to solve it?
2. How did you solve the second task? What competence did you need to have to solve it?
3. Which of the two tasks was more difficult? What was difficult?
4. How do the processes of solving each task differ? In what way are they similar?

The analysis of the written work of students was to find the answer to the following research questions:

1. Do the students see in these tasks the elements of widely understood mathematical modelling?
   - Do the respondents notice the difference between the types of tasks?
   - Are the respondents able to explain the difference between the approaches to the discussed tasks?
   - Are they able to name the skills needed to solve them?
   - Can they notice non-mathematical (non-computational) competence needed in exc.1?
2. How does the process of creating the model by the students look like?

- Is the cyclical nature of this process visible?
- Do the students check the results?
- Do they see the necessity of the analysis of the information in exc.1?

I expected that the students would see the difference in the didactic approach to solve the tasks. I hoped they would see that the second task needed only the application of the indicated mathematical tool, whereas the first one involved creating or selecting a mathematical model of the situation shown. I also expected that the respondents, while describing the skills used, would associate them with the competence related to mathematical modelling.

In order to recognize the students' knowledge on mathematical modelling, I asked the following specific questions:

- Are the future teachers familiar with the term "mathematical modelling"?
- What do the students understand by mathematical modelling?
- Do they see mathematical modelling in school education?
- Are the students aware of when mathematical modelling appears in school education?
- Do the future teachers know that mathematical modelling belongs to one of the main groups of teaching mathematics objectives?
- Are the students able to list the forms that a model can take?
- Are the respondents aware that modelling is a process?
- Do they know the stages of the modelling process?

I searched for the answers to these questions by analysing the students' statements from the questionnaire:

5. *How do you understand "mathematical modelling"?*
6. *In what form can a mathematical model be presented?*
7. *Where in school education does mathematical modelling appear?*
8. *What stages of the modelling process can you name?*

I was also interested in whether the students could see the relation between the questions from both parts of the questionnaire or whether questions related to modelling would suggest some relation and influence the answers for the questions from the first part of the questionnaire.

5.1.2. *Organization of research.* The research was conducted in January 2014 as part of classes on didactics of mathematics of the fourth stage of education. The questionnaire was completed by 12 second-year maths students of complementary master's studies with specialization in teaching mathematics and computer science. Time to complete the tasks was adjusted to the needs of the students and took about 30 minutes.

5.2. **The analysis of the respondents' work.**

5.2.1. *The analysis of the approach to solving the tasks and answering the questions from the questionnaire (1-4).* Two tasks constituting a part of the research tool were not difficult from the factual point of view – all the respondents did them correctly. It turned out, however, that they are significantly difficult from the didactic perspective. Although the research was carried out on future teachers, none of them recognized that the first of the two tasks requires an unassisted discovery of the research tool, which is understood as mathematical modelling. The students do not associate this task with mathematical modelling that requires: the analysis of the situation and the information given, selection of the data and evaluation of their relevancy, thinking over the way to solve it, bringing a real situation to the level of mathematical calculations by creating an appropriate model (proportions or equations in this case) and verification of the results. They focus only on computational solution and not on didactic aspects of solving the task. The respondents cannot show the difference between these tasks. None of them noticed that in the first task one has to choose the method of solving the task on their own, whereas in the second task it was explicitly stated that one has to use percentage calculations. Only one person noticed that the second task is purely a mathematical computational task in which non-mathematical content is irrelevant. He or she writes that the difference may lie in the fact that the first task is a typical instructional task that is based on the correct understanding, whereas the second one focuses only on calculations. It is not true, though, that the first task is a typical instructional task. Such perception of the task indicates traditional computational techniques (verification of the data, the unknowns, calculations), which may suggest that the students associate solving tasks more with performing calculations than with creating a model. Unfortunately, focusing only on the methods of calculation does not result in the ability to smooth mathematical modelling (see: Niss). However, the author of the above statement emphasizes the fact that in the second task the whole activity is reduced only to the correct application of a known and suggested calculation. This statement indicates that despite the lack of clear differentiation between the types of the tasks, students intuitively perceive the difference in the approaches to them. The statements of 10 persons suggest different attitudes towards considered problems. Although they cannot explain it, 8 persons say that the first task was more difficult, whereas 3 persons write that the second task can be solved in a mental calculation. The intuition of the respondents is visible in the amount of description – they pay more attention to task 1, they provide more sophisticated responses in the questionnaire or write down calculations and proportions, whereas in task 2 there are often only the results. The intuitive differentiation between the types of tasks is suggested in such statements as: "The first task was more difficult because one had to think longer about the instruction and carefully analyse the table so as to create a well-arranged proportion". "In the first task, one had to think more about how to arrange proportions well, whereas in the second task one had only to calculate the percentage of a given number". The underlined words suggest that the respondents felt that task 1 was more demanding – it was more time-consuming and involved more analysis of the data and thinking about the problem in it, creating the tool was more difficult, the correctness of this tool is not obvious, whereas in task 2 one had "only to calculate" – one did not have to think but to do the calculations mechanically. The statements of 7 persons suggest that they see in task 1 the necessity of the analysis of the information given – as required competence here they indicate the ability to read with understanding and logical thinking. In none of the works checking the results appears, but in two of them various elements were used to solve the task (fat, calcium, energy), which may indicate the attempt to verify the results. As a way of solving the tasks, the students indicate the use of proportion and the calculation of the percentage of a given number. They consider the knowledge of the above methods, as well as reading with understanding, logical thinking, knowledge of multiplication tables or the ability to convert units, as necessary competence, however none of them noticed in task 1 non-mathematical abilities, which Maaß, Perrenet and Zwaneveld point to.

The conducted analysis shows that the students, although they intuitively see the difference in the approach to the tasks, cannot formulate this and focus on calculations, not on teaching benefits that can be obtained from these tasks.

5.2.2. *The analysis of the part of the questionnaire related to the knowledge about mathematical modelling.* Future teachers should be familiar with the concept of mathematical modelling to teach their future students abilities related to it. The analysis of the students' written responses shows that their knowledge about modelling is little. I suppose this is because teachers do not attach importance to the very concept of modelling in the course of teaching. The skills related to it (according to teaching mathematics objectives in the core curriculum) should be taught at every stage of education and so it happens, however important information is usually overlooked that these activities aim at developing skills of mathematical modelling. Consequently, there appears lack of understanding of the concept of modelling, which can be confirmed in the students statements from the questionnaire. The analysis of the works showed that among 12 respondents only 5 of them show understanding of mathematical modelling that is close to its real meaning, of which 2 of these definition can be considered correct, e.g. “By mathematical modelling I understand a process of creating a pattern (searching for arithmetical dependencies) between certain variables (parameters) so as to simplify future calculations by application of data”. The majority of the students (7 persons) associate mathematical modelling with creating a pattern, a formula, an equation or a function, which is a good association, but they are not able to precisely define what this term means. Unfortunately, there are also answers completely far from the truth, e.g. “I understand mathematical modelling as the way in which the teacher guides the student to the correct way to solve the task”. In the interview conducted after completing the questionnaire the students admitted that they do not exactly know what mathematical modelling is but for 2 people who came across this concept on different studies.

Despite problems with explaining what mathematical modelling is, the majority of the respondents (9 persons) can see it in school education, however it is difficult for them to state where exactly it appears. Very general answers appear – on computer science, mathematical and natural subjects, during studies and in high school. Three of these answers are wrong and it results from wrong understanding of modelling, e.g. “on every maths class” – said one of the students who perceives modelling as kind of the teacher’s gun guiding a student to the correct solution. Some generally formulated answers also appear which can be considered correct, e.g. functions, planimetry, stereometry. In none of the students’ statements reference to the objectives of teaching mathematics appears, although mathematical modelling belongs to one of the groups of these objectives.

A mathematical model can take various forms and the students’ answers prove this. Although only 6 persons answered the question about the form of a model, the majority of them were correct. Most frequently mentioned form was that of the graph and the equation with unknowns, but also such forms were mentioned as the description, diagram, inequality, block diagram. Among the incorrect answers were the example and the word problem.

Three persons emphasized that mathematical modelling is a process but none of them explained exactly why – students treat modelling as a single activity, without the necessity to test or improve the model. Four students made an attempt to list the stages of the modelling process. However, these concern only a single creation of the model and not the process of improvement of the working model. The mentioned stages are the following: choosing the problem, analysing the task, collecting the data, searching for and writing dependencies between the variables, solving equations. The student who mentioned the most stages was the one to mention the most of them and he or she described the process of creating a model the most precisely, however with limiting it down to creating only the first version of the model.

The conducted research reveals students' lack of appropriate knowledge in the field of mathematical modelling although they managed to do the tasks, one of which required the use of mathematical modelling. Only in two out of twelve works we can see that the respondent, while answering the questions related to modelling, tries to relate to the tasks done at the beginning. In the rest of works no dependencies between these two parts of the questionnaire appear.

6. Summary

Mathematical modelling is a concept that covers a wide range of activities. It is differently perceived by various groups of users depending on which activity they focus on. Mathematical modelling can be understood both as formulation of an equation, a function, etc., describing a given situation and as a cyclical process of creating a model that balances between the real world and the mathematical world.

Future teachers of mathematics have little knowledge on this subject. They cannot name the differences between a task that requires the skills related to mathematical modelling and a typical computational task, however their answers suggest that they intuitively see the difference in the approach to these tasks. Therefore, it appears that we should not delude ourselves that students will gain non-mathematical competence related to mathematical modelling but we have to explicitly show them the difference in the approach towards non-computational tasks which require the analysis of the real situation and where a decisive factor concerning, inter alia, the selection of information is important. One fact is comforting (as confirmed by researchers in the field of mathematical modelling) that one can effectively teach these skills that lie beyond pure mathematics, which can result in numerous didactic benefits.

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Received: May 2014

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