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BOUNDARY VALUE PROBLEMS FOR POISSON INTEGRALS FOR HERMITE EXPANSIONS

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Abstract

The aim of this paper is the study the Poisson integral for Hermite expansions. We present some boundary value problems related to this integral and its various modifications.

1. Introduction

Let $L^p(\mathbb{R})$ denote the set of functions f defined on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |f(t)|^p \, dt < \infty \quad \text{if } 1 \le p < \infty,$$

and f is bounded almost everywhere on $\mathbb{R}$ if $p = \infty$.

In the paper [4] the author presented some approximation properties of the Poisson integral for Hermite function expansions given by

$$A(f)(r,y) = A(f;r,y) = \int_{-\infty}^{\infty} r^{\frac{1}{2}} K(r,y,z)\, f(z)\, dz, \quad f \in L^p(\mathbb{R}),$$

where

$$K(r,y,z) = \sum_{n=0}^{\infty} r^n h_n(y)\, h_n(z), \quad 0 < r < 1,$$

$$h_n(x) = \left(2^n n! \sqrt{\pi}\right)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right) H_n(x)$$

and H_n is the nth Hermite polynomial (see, for example, [10]). The operator $A(f)$ is linear and positive. Basic facts on positive linear operators and its applications can be found in [1, 2].

In [4] the following theorem was proved.

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Theorem 1. [4] *Let* $y_0 \in \mathbb{R}$ *and let* $f = f_1 + f_2$*, where* $f_1 \in L^1(\mathbb{R})$*,* $f_2 \in L^\infty(\mathbb{R})$*. If* f *is continuous at* y_0*, then*

$$\lim_{(r,y)\to(1^-,y_0)} A(f;r,y) = f(y_0).$$

Gosselin and Stempak in [3] considered the integral $A_0(f)$ of a function $f \in L^p(\mathbb{R})$ defined by

$$A_0(f)(x,y) = A_0(f;x,y) = \int_{-\infty}^{\infty} P(x,y,z)f(z)\,dz,$$

where

$$P(x,y,z) = \sum_{n=0}^{\infty} h_n(y)\,h_n(z)\exp\left(-(2n+1)x\right), \quad x > 0$$

and

$$P(x,y,z) = e^{-x}K(e^{-2x},y,z).$$

Gosselin and Stempak [3] obtained the following results.

Theorem 2. [3] *If* $f \in L^p(\mathbb{R})$*,* $1 \le p \le \infty$*, then* $A_0(f)$ *is of the class* C^∞ *on the set* $(0,\infty)\times\mathbb{R}$ *and* $A_0(f)$ *is a solution of the differential equation*

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial^2 u(x,y)}{\partial y^2} - y^2 u(x,y).$$

Theorem 3. [3] *Let* $f \in L^p(\mathbb{R})$*. Then*

(a) $\|A_0(f;x,\cdot)\|_p \le (\cosh 2x)^{-\frac{1}{2}}\|f\|_p, \quad 1 \le p \le \infty,$
(b) $\|A_0(f;x,\cdot) - f(\cdot)\|_p \to 0$ *as* $x \to 0$, $1 \le p < \infty$,
(c) $\lim_{x\to 0} A_0(f;x,y) = f(y)$ *almost everywhere,* $1 \le p < \infty$.

It is worth to mention that approximation properties of various Poisson integrals associated with Hermite and Laguerre polynomials were studied in one and two dimensions in [5, 6, 7, 8, 9, 11].

In this paper we indicate boundary value problems related to $A(f)$ and some modifications of this operator.

2. Boundary value problems

Below we present announced theorems. We omit the proofs of them, because there are a simple consequence of previous properties.

Theorem 4. *Let* $f \in L^p(\mathbb{R})$*,* $1 \le p \le \infty$*. Then* $A(f)$ *is of the class* C^∞ *on the set* $(0,1)\times\mathbb{R}$*. Moreover,* $A(f)$ *is a solution of the problem*

$$-2r\frac{\partial u(r,y)}{\partial r} = \frac{\partial^2 u(r,y)}{\partial y^2} - y^2 u(r,y), \qquad (r,y) \in (0,1)\times\mathbb{R},$$

$$\lim_{r\to 1^-} \|u(r,\cdot) - f(\cdot)\|_p = 0, \qquad 1 \le p < \infty.$$

We introduce the operator A_1 given by

$$A_1(f)(t,y) = A_1(f;t,y) = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{t}\right) K\left(\exp\left(-\frac{2}{t}\right), y, z\right) f(z)\, dz$$

for $f \in L^p(\mathbb{R}),\ 1 \le p \le \infty,\ t > 0$ and $y \in \mathbb{R}$.

Theorem 5. *Let $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$. Then $A_1(f)$ is of the class C^∞ on the set $\mathbb{R}_+ \times \mathbb{R}$ and $A_1(f)$ is a solution of the problem*

$$-t^2 \frac{\partial u(t,y)}{\partial t} = \frac{\partial^2 u(t,y)}{\partial y^2} - y^2 u(t,y), \qquad (t,y) \in \mathbb{R}_+ \times \mathbb{R},$$

$$\lim_{t\to\infty} \|u(t,\cdot) - f(\cdot)\|_p = 0, \qquad 1 \le p < \infty.$$

Let us consider the operator A_2 defined by

$$A_2(f)(r,y) = A_2(f;r,y) = \rho(r) \int_{-\infty}^{\infty} K(r,y,z) f(z)\, dz$$

for $f \in L^p(\mathbb{R}),\ 1 \le p \le \infty,\ 0 < r < 1$, where the function ρ is continuously differentiable in $(0,1)$ and such that

$$\rho(r) > 0 \quad \text{and} \quad \lim_{r\to 1^-} \rho(r) = 1.$$

We introduce the notation

$$T = \frac{\partial^2}{\partial y^2} - y^2 + 2r\frac{\partial}{\partial r} - 2r\frac{\rho'(r)}{\rho(r)} + 1 \quad \text{and} \quad T^2 = T(T).$$

Theorem 6. *Let $y_0 \in \mathbb{R}$. If f is as in Theorem 1, then $A_2(f)$ is of the class C^∞ on the set $(0,1) \times \mathbb{R}$ and $A_2(f)$ is a solution of the problem*

$$Tu(r,y) = 0, \quad (r,y) \in (0,1) \times \mathbb{R},$$

$$\lim_{(r,y)\to(1^-,y_0)} u(r,y) = f(y_0).$$

For $f, g \in L^p(\mathbb{R}),\ 1 \le p \le \infty$ we define the operator V:

$$V(f,g)(r,y) = V(f,g;r,y) = \rho_1(r) A_2(f;r,y) + A_2(g;r,y),$$

where the function ρ_1 is continuously differentiable in $(0,1)$, $0 < r < 1$, $y \in \mathbb{R}$.

Theorem 7. *Let $y_0 \in \mathbb{R}$. If f, g are as in Theorem 1 and*

$$\lim_{r\to 1^-} \rho_1(r) = 0, \quad \lim_{r\to 1^-} \rho_1'(r) = \frac{1}{2}, \quad \frac{\partial}{\partial r}\left(r\rho_1'(r)\right) = 0,$$

then $V(f,g)$ is of the class C^∞ on the set $(0,1)\times\mathbb{R}$ and $V(f,g)$ is a solution of the problem

$$T^2u(r,y) = 0, \quad (r,y) \in (0,1)\times\mathbb{R},$$
$$\lim_{(r,y)\to(1^-,y_0)} u(r,y) = g(y_0),$$
$$\lim_{(r,y)\to(1^-,y_0)} Tu(r,y) = f(y_0).$$

Remark 1. *From the assumptions of Theorem 7 it follows that $\rho_1(r) = \frac{1}{2}\ln r$. In this case the operator V is of the form*

$$V(f,g;r,y) = \frac{1}{2}\rho(r)\ln r \int_{-\infty}^{\infty} K(r,y,z)f(z)\,dz + \rho(r)\int_{-\infty}^{\infty} K(r,y,z)g(z)\,dz$$

for $0<r<1$, $y\in\mathbb{R}$.

Theorem 8. *Let $y_0 \in \mathbb{R}$. If f, g are as in Theorem 1 and if*

$$\lim_{r\to 1^-} \rho_1(r) = 0, \quad \lim_{r\to 1^-} \rho_1'(r) = \frac{1}{2},$$
$$2r\frac{\partial}{\partial r}\left(r\rho_1'(r)\right) + r\rho_2(r)\rho_1'(r) = 0,$$

where ρ_2 is some continuous function, then $V(f,g)$ is of the class C^∞ on the set $(0,1)\times\mathbb{R}$ and $V(f,g)$ is a solution of the problem

$$T^2u(r,y) + \rho_2(r)Tu(r,y) = 0, \quad (r,y) \in (0,1)\times\mathbb{R},$$
$$\lim_{(r,y)\to(1^-,y_0)} u(r,y) = g(y_0),$$
$$\lim_{(r,y)\to(1^-,y_0)} Tu(r,y) = f(y_0).$$

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