Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XIX (2014)
227–235

K-CONTINUITY PROBLEM OF K-SUPERQUADRATIC SET-VALUED FUNCTIONS

KATARZYNA TROCZKA-PAWELEC

Abstract

In this paper we study K-superquadratic set-valued functions.We will present here some connections between K-boundedness of K-superquadratic set-valued functions and K-semicontinuity of multifunctions of this kind.

1. Introduction

Let $X = (X, +)$ be an arbitrary topological group. A real-valued function F is called superquadratic, if it fulfils inequality

$$2F(x) + 2F(y) \leq F(x + y) + F(x - y), \quad x, y \in X. \tag{1}$$

If the sign "$\leq$" in (1) is replaced by "$\geq$", then F is called subquadratic. The continuity problem of functions of this kind was considered in [2]. This problem was also considered in the class of set-valued functions. In this case F is called subquadratic set-valued function, if it satisfies inclusion

$$F(x + y) + F(x - y) \subset 2F(x) + 2F(y), \quad x, y \in X \tag{2}$$

and superquadratic set-valued function, if it satisfies inclusion defined in such a form:

$$2F(x) + 2F(y) \subset F(x + y) + F(x - y), \quad x, y \in X. \tag{3}$$

For usual (i.e. single-valued) functions the properties of subquadratic and superquadratic functions are quite analogous and, in view of the fact that if a function F is subquadratic, then the function $-F$ is superquadratic and conversely, it is not necessary to investigate functions of these two kinds individually.

In the case of set-valued functions the situation is different. Even if properties of subquadratic and superquadratic set-valued functions are similar, we have to prove them separately.

• *Katarzyna Troczka-Pawelec* — e-mail: k.troczka@ajd.czest.pl
Jan Długosz University in Częstochowa.

If the sign "$\subset$" in the inclusions above is replaced by "$=$", then F is called quadratic set-valued function. The class of quadratic set-valued functions is an important subclass of the class of subquadratic and superquadratic set-valued functions. Quadratic set-valued functions have already extensive bibliography (see D. Henney [1], K. Nikodem [4] and W. Smajdor [5]). The continuity problem of subquadratic and superquadratic set-valued functions was considered in [6] and [7].

If we enlarge the space of values of a set-valued function F by a cone K we can consider K-superquadratic set-valued functions, that is solutions of the inclusion

$$F(x+y)+F(x-y)\subset 2F(x)+2F(y)+K,\qquad x,y\in X. \tag{4}$$

The concept of K-superquadraticity is related to real-valued superquadratic functions. Note, in the case when F is a single-valued real function and $K=[0,\infty)$, we obtain the standard definition of superquadratic functions (1).

Similarly, if a set-valued function F satisfies the following inclusion

$$2F(x)+2F(y)\subset F(x+y)+F(x-y)+K,\qquad x,y\in X \tag{5}$$

then it is called K-subquadratic. The K-continuity problem of multifunction of this kind was considered in [8]. It has been proved there that a K-subquadratic set-valued function F defined on 2-divisible topological group X with non-empty, compact and convex values in a locally convex topological vector space Y, which is K-continuous at zero and locally K-bounded in X, is K-continuous everywhere in X.

In this paper we shall consider similar problem for K-superquadratic set-valued functions. Likewise as in functional analysis we can look for connections between K-boundedness and K-semi-continuity of set-valued functions of this kind.

Assuming $K=\{0\}$ in (4) and (5), we obtain the inclusions (2) and (3).

Let us start with the notations used in this paper. Let Y be a topological vector space. Let $n(Y)$ denotes the family of all non-empty subsets of Y and $cc(Y)$−the family of all compact and convex members of $n(Y)$. The term *set-valued function* will be abbreviated to the form s.v.f.

Recall that a set $K\subset Y$ is called a cone iff $K+K\subset K$ and $sK\subset K$ for all $s\in(0,\infty)$.

Definition 1. (cf. [3]) *A cone K in a topological vector space Y is said to be a normal cone iff there exists a base $\mathfrak{W}$ of zero in Y such that*

$$W=(W+K)\cap(W-K)$$

for all $W\in\mathfrak{W}$.

Definition 2. (cf. [3]) *An s.v.f.* $F\colon X \to n(Y)$ *is said to be* K*-upper semicontinuous (abbreviated* K*-u.s.c.) at* $x_0 \in X$ *iff for every neighbourhood* V *of zero in* Y *there exists a neighbourhood* U *of zero in* X *such that*

$$F(x) \subset F(x_0) + V + K$$

for every $x \in x_0 + U$.

Definition 3. (cf. [3]) *An s.v.f.* $F\colon X \to n(Y)$ *is said to be* K*-lower semicontinuous (abbreviated* K*-l.s.c.) at* $x_0 \in X$ *iff for every neighbourhood* V *of zero in* Y *there exists a neighbourhood* U *of zero in* X *such that*

$$F(x_0) \subset F(x) + V + K$$

for every $x \in x_0 + U$.

Definition 4. (cf. [3]) *An s.v.f.* $F\colon X \to n(Y)$ *is said to be* K*-continuous at* $x_0 \in X$ *iff it is both* K*-u.s.c. and* K*-l.s.c. at* x_0. *It is said to be* K*-continuous iff it is* K*-continuous at each point of* X.

Note that in the case where $K = \{0\}$ the K-continuity of F means its continuity with respect to the Hausdorff topology on $n(Y)$.

In this paper we will use the following lemma.

Lemma 1. (cf. [8]) *Let* Y *be a topological vector space and* K *be a cone in* Y. *Let* A, B, C *be non-empty subsets of* Y *such that* $A+C \subset B+C+K$. *If* B *is convex and* C *is bounded, then* $A \subset \overline{B+K}$.

2. The main result

In the proof of the main theorem we will often use four known lemmas (see Lemma 1.1, Lemma 1.3, Lemma 1.6 and Lemma 1.9 in [9]). The first lemma says that for a convex subset A of an arbitrary real vector space Y the equality $(s+t)A = sA + tA$ holds for every $s, t \geq 0$ or (s,t<0). The second lemma says that in a real vector space Y for two convex subsets A, B the set $A + B$ is also convex. The next lemma says that if $A \subset Y$ is a closed set and $B \subset Y$ is a compact set, where Y denotes a real topological vector space, then the set $A + B$ is closed. For any sets $A, B \subset Y$, where Y denotes the same space as above, the inclusion $\overline{A} + \overline{B} \subset \overline{A+B}$ holds and the equality holds if and only if the set $\overline{A} + \overline{B}$ is closed.

Notice that for the cone K the following remark holds.

Remark 1. *Let* Y *be a real topological vector space. If* K *is a closed cone, then it is a cone with zero.*

Let us adopt the following three definitions which are natural extension of the concept of the boundedness for real-valued functions.

Definition 5. *An s.v.f.* $F\colon X \to n(Y)$ *is said to be* K*-lower bounded on a set* $A \subset X$ *iff there exists a bounded set* $B \subset Y$ *such that* $F(x) \subset B + K$ *for all* $x \in A$*. An s.v.f.* $F\colon X \to n(Y)$ *is said to be* K*-lower bounded at a point* $x \in X$ *iff there exists a neighbourhood* U_x *of zero in* X *such that* F *is* K*-lower bounded on a set* $x + U_x$

Definition 6. *An s.v.f.* $F\colon X \to n(Y)$ *is said to be* K*-upper bounded on a set* $A \subset X$ *iff there exists a bounded set* $B \subset Y$ *such that* $F(x) \subset B - K$ *for all* $x \in A$*. An s.v.f.* $F\colon X \to n(Y)$ *is said to be* K*-upper bounded at a point* $x \in X$ *iff there exists a neighbourhood* U_x *of zero in* X *such that* F *is* K*-upper bounded on a set* $x + U_x$

Definition 7. *An s.v.f.* $F\colon X \to n(Y)$ *is said to be locally* K*-bounded in* X *iff it is both* K*-upper and* K*-lower bounded at every point* $x \in X$*.*

Definition 8. *We say that* 2*-divisible topological group* X *has the property* $\left(\frac{1}{2}\right)$ *iff for every neighbourhood* V *of zero there exists a neighbourhood* W *of zero such that* $\frac{1}{2}W \subset W \subset V$*.*

For the K-superquadratic set-valued functions the following theorem holds.

Theorem 1. *Let* X *be a* 2*-divisible topological group with property* $\left(\frac{1}{2}\right)$*,* Y *– locally convex topological real vector space and* $K \subset Y$ *a closed normal cone. If a* K*-superquadratic s.v.f.* $F\colon X \to cc(Y)$ *is* K*-u.s.c. at zero,* $F(0) = \{0\}$ *and locally* K*-bounded in* X*, then it is* K*-u.s.c. in* X*.*

Proof. Suppose that F is not K-u.s.c. at a point $z \in X$, i.e. there exists a neighbourhood V of zero in Y such that for every neighbourhood U of zero in X we can find $x_u \in U$ for which

$$F(z + x_u) \not\subseteq F(z) + V + K.$$

Take a balanced convex neighbourhood W of zero in Y such that

$$W \subset V$$

and

$$\overline{F(z) + W + K} \subset F(z) + V + K.$$

Then also

$$F(z + x_u) \not\subseteq \overline{F(z) + W + K}. \tag{6}$$

Let a neighbourhood U of zero in X be arbitrarily fixed. Suppose that

$$F(z + x_u) + 2^k \left(2^k - 1\right) F(x_u) \not\subseteq \overline{F\left(z + \left(1 - 2^k\right) x_u\right) + 2^k W + K} \tag{7}$$

for some $k \in \mathbb{N} \cup \{0\}$. The proof of (7) runs by induction. For $k = 0$ condition (7) holds with respect to (6). Putting $y = x$ in (4) and using condition $F(0) = \{0\}$, we have

$$F(2x) \subset 4F(x) + K.$$

An easy induction shows

$$F(2^n x) \subset 4^n F(x) + K \tag{8}$$

for $x \in X$ and for all positive integers $n \in \mathbb{N}$. By K-superquadraticity of F and (8), we have

$$\begin{aligned} &F\left(z + (1 - 2^{k+1})x_u\right) + F(z + x_u) = \\ &= F\left(z + x_u - 2^k x_u - 2^k x_u\right) + F\left(z + x_u - 2^k x_u + 2^k x_u\right) \subset \\ &\subset 2F\left(z + x_u - 2^k x_u\right) + 2F\left(2^k x_u\right) + K \subset \\ &\subset 2F\left(z + (1 - 2^k)x_u\right) + 2^{2k+1} F(x_u) + K. \end{aligned} \tag{9}$$

In view of the fact that for any sets $A, B \subset Y, \overline{A} + \overline{B} \subset \overline{A + B}$ we get

$$\begin{aligned} &\overline{F\left(z + (1 - 2^k)x_u\right) + 2^k W + K} + K \subset \\ &\subset \overline{F\left(z + (1 - 2^k)x_u\right) + 2^k W + K} \end{aligned}$$

and, consequently,

$$\begin{aligned} &\overline{\overline{F(z + (1 - 2^k)x_u) + 2^k W + K} + K} \subset \\ &\subset \overline{F(z + (1 - 2^k)x_u) + 2^k W + K}. \end{aligned} \tag{10}$$

By (7) and (10), we obtain

$$F(z + x_u) + 2^k \left(2^k - 1\right) F(x_u) \not\subseteq \overline{\overline{F(z + (1 - 2^k)x_u) + 2^k W + K} + K}.$$

Notice that for a cone K the equality $aK = K$ holds for every $a \in (0, \infty)$. Hence,

$$\begin{aligned} &2F(z + x_u) + 2^{k+1} \left(2^k - 1\right) F(x_u) \not\subseteq \\ &\not\subseteq \overline{\overline{2F(z + (1 - 2^k)x_u) + 2^{k+1} W + K} + K}. \end{aligned} \tag{11}$$

By (11) and Lemma 1,

$$\begin{aligned} &2F(z + x_u) + 2^{k+1} \left(2^k - 1\right) F(x_u) + 2^{2k+1} F(x_u) \not\subseteq \\ &\not\subseteq \overline{2F(z + (1 - 2^k)x_u) + 2^{k+1} W + K} + 2^{2k+1} F(x_u) + K. \end{aligned}$$

In view of Remark 1, K is a cone with zero. Therefore by above,

$$2F(z+x_u)+2^{k+1}\left(2^k-1\right)F(x_u)+2^{2k+1}F(x_u)+K\not\subseteq \tag{12}$$

$$\not\subseteq \overline{2F(z+(1-2^k)x_u)+2^{k+1}W+K}+2^{2k+1}F(x_u)+K.$$

In view of the fact that the sum of closed and compact sets is closed and for any sets $A,B\subset Y, \overline{A}+\overline{B}=\overline{A+B}$, in the case where $\bar{A}+\bar{B}$ is a closed set, we get

$$\overline{2F(z+(1-2^k)x_u)+2^{k+1}W+K}+2^{2k+1}F(x_u)= \tag{13}$$

$$=\overline{2F(z+(1-2^k)x_u)+2^{k+1}W+K+2^{2k+1}F(x_u)}.$$

Since K is a cone, by (9), we obtain

$$\overline{F(z+(1-2^{k+1})x_u)+F(z+x_u)+2^{k+1}W+K}\subset \tag{14}$$

$$\subset\overline{2F(z+(1-2^k)x_u)+2^{k+1}W+K+2^{2k+1}F(x_u)}.$$

Since F has closed values, we get

$$F(z+x_u)+\overline{F(z+(1-2^{k+1})x_u)+2^{k+1}W+K}+K\subset \tag{15}$$

$$\subset\overline{F(z+(1-2^{k+1})x_u)+F(z+x_u)+2^{k+1}W+K}+K.$$

Consequently, by (12–15) we conclude

$$2F(z+x_u)+2^{k+1}\left(2^k-1\right)F(x_u)+2^{2k+1}F(x_u)+K\not\subseteq$$

$$\not\subseteq F(z+x_u)+\overline{F(z+(1-2^{k+1})x_u)+2^{k+1}W+K}+K.$$

By convexity of the sets $F(x_u)$ i $F(z+x_u)$, we obtain

$$F(z+x_u)+F(z+x_u)+2^{k+1}\left(2^{k+1}-1\right)F(x_u)+K\not\subseteq$$

$$\not\subseteq F(z+x_u)+\overline{F(z+(1-2^{k+1})x_u)+2^{k+1}W+K}+K.$$

Therefore,

$$F(z+x_u)+2^{k+1}\left(2^{k+1}-1\right)F(x_u)\not\subseteq$$

$$\not\subseteq\overline{F(z+(1-2^{k+1})x_u)+2^{k+1}W+K}.$$

We have proved that (7) holds for every neighbourhood U of zero in X and $k=0,1,2...$

Since K is a normal cone, there exists a base $\mathfrak{W}$ of neighbourhoods of zero in Y such that $M=(M+K)\cap(M-K)$ for all $M\in\mathfrak{W}$. We can choose $W_1\in\mathfrak{W}$ and balanced neighbourhood W_2 of zero in Y such that

$$W_2\subset W_1\subset W.$$

Because F is K-lower bounded on a neighbourhood of z, there exists a neighbourhood U_0 of zero in X and a bounded set $B_1 \subset Y$ such that

$$F(z+t) \subset B_1 + K, \qquad t \in U_0.$$

Since the set B_1 is bounded, there exists $\lambda_1 > 0$ such that

$$B_1 \subset \frac{1}{\lambda_1} W_2.$$

Therefore, by above,

$$F(z+t) \subset \frac{1}{\lambda_1} W_2 + K, \qquad t \in U_0.$$

Similarly, since F is K-upper bounded on a neighbourhood of z, there exists a neighbourhood U_1 of zero in X and a bounded set $B_2 \subset Y$ such that

$$F(z+t) \subset B_2 - K, \qquad t \in U_1.$$

Since the set B_2 is bounded, there exists $\lambda_2 > 0$ such that

$$B_2 \subset \frac{1}{\lambda_2} W_2.$$

Therefore, by above,

$$F(z+t) \subset \frac{1}{\lambda_2} W_2 - K, \qquad t \in U_1.$$

Let $\lambda := \min\{\lambda_1, \lambda_2\}$. Since W_2 is a balanced set, we get

$$F(z+t) \subset \frac{1}{\lambda} W_2 + K \subset \frac{1}{\lambda} W_1 + K, \qquad t \in U_0 \tag{16}$$

and

$$F(z+t) \subset \frac{1}{\lambda} W_2 - K \subset \frac{1}{\lambda} W_1 - K, \qquad t \in U_1. \tag{17}$$

By (16) and (17), we obtain

$$F(z+t) \subset \left(\frac{1}{\lambda} W_1 + K\right) \cap \left(\frac{1}{\lambda} W_1 - K\right), \qquad t \in U_0 \cap U_1. \tag{18}$$

Because of $W_1 \in \mathfrak{W}$, we have

$$\left(\frac{1}{\lambda} W_1 + K\right) \cap \left(\frac{1}{\lambda} W_1 - K\right) = \frac{1}{\lambda} W_1$$

and, consequently, the following inclusion holds

$$F(z+t) \subset \frac{1}{\lambda} W \tag{19}$$

for every $t \in U_0 \cap U_1$.

Let $k \in \mathbb{N}$ be so large that

$$2^k > \frac{3}{\lambda}. \tag{20}$$

Let U be a symmetric neighbourhood of zero in X such that $U+U \subset U_0 \cap U_1$ and $\frac{1}{2}U \subset U$. Consider two sets $\frac{1}{2^k}U$ i $\frac{1}{\lambda 2^k(2^k-1)}W$. Since F is K-u.s.c. at zero and $F(0) = \{0\}$, there exists a symmetric neighbourhood U_2 of zero in X such that

$$U_2 \subset \frac{1}{2^k}U \subset U \tag{21}$$

and

$$F(t) \subset \frac{1}{\lambda 2^k(2^k-1)}W + K, \qquad t \in U_2. \tag{22}$$

There exists $x_u \in U_2$ such that (7) holds. By (21),

$$\left(1 - 2^k\right) x_u = x_u - 2^k x_u \in U_2 - U \subset U + U \subset U_0 \cap U_1 \tag{23}$$

and by (22),

$$F(x_u) \subset \frac{1}{\lambda 2^k(2^k-1)}W + K. \tag{24}$$

Let $a \in F\left(z + (1-2^k)x_u\right)$, $b \in F(z+x_u)$ i $c \in F(x_u)$. By (19), (20), (23) and (24), we obtain

$$b + 2^k\left(2^k - 1\right)c - a \in \frac{1}{\lambda}W + \frac{1}{\lambda}W + K + \frac{1}{\lambda}W \subset 2^k W + K.$$

Therefore,

$$b + 2^k\left(2^k - 1\right)c \in F\left(z + (1-2^k)x_u\right) + 2^k W + K.$$

We have proved that

$$F(z+x_u) + 2^k\left(2^k - 1\right)F(x_u) \subset F\left(z + (1-2^k)x_u\right) + 2^k W + K,$$

which contradicts (7). □

This article is an introduction to the discussion on the K-continuity problem for K-superquadratic set-valued functions. In the theory of K-subquadratic and K-superquadratic set-valued functions an important role is played by theorems giving possibly weak conditions under which such multi-functions are K-continuous.

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Received: October 2014

K. Troczka-Pawelec
Jan Długosz University in Częstochowa,
Institute of Mathematics and Computer Science,
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
E-mail address: k.troczka@ajd.czest.pl