Jan Długosz University of Częstochowa
Scientific Issues, Mathematics XI, Częstochowa 2006

Queueing Systems with Random Volume Internal and External Calls

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Abstract In the present paper we investigate a single-server $BM/G/1/\infty$ queueing system with non-homogeneous calls of two following types: 1) external calls served by the system under consideration, 2) internal calls arrive only when an external call is served and interrupts the service process. The external calls appear according to a stationary Poisson process with bulk arrivals. Calls of each from above-mentioned types are characterized by some random volume. Service time of the call arbitrarily depends on its volume. Two schemes of calls service organization are analyzed. The non-stationary and stationary total calls volume distribution is determined in terms of Laplace and Laplace-Stieltjes transforms. The stationary first moment of total calls volume distribution is calculated for each scheme.

1. Introduction

We consider $BM/G/1/\infty$ queueing system with calls of two following types: 1) external calls, 2) internal calls.

Extended version of a talk presented at the X Conference "Applications of Algebra in Logic and Computer Science", Zakopane, March 6–12, 2006.

External calls appear according to a stationary Poisson process with bulk arrivals. It means that epochs of calls arriving make up a stationary Poisson process with rate a, and a group consisting of k, $k = 1, 2, \ldots,$ calls arrives at such epoch with probability g_k, $\sum\limits_{k=1}^{\infty} g_k = 1$, independently of the arriving time moment and characteristics of another groups. So the generation function $G(z) = \sum\limits_{k=1}^{\infty} g_k z^k$ of the number of calls in the group is known. An external call is characterized by some random volume ζ, its service time ξ can be dependent on the volume. Let $F(x,t) = \mathbf{P}\{\zeta < x, \xi < t\}$ be the joint distribution function of random variables ζ and ξ, $L(x) = F(x, \infty)$ and $B(t) = F(\infty, t)$ be the distribution functions of random variables ζ and ξ, respectively.

Internal calls arrive only during external calls service. If T is an epoch of external call service beginning and the service process is not completed in the interval $[T; T+t)$, an internal call appears in this time interval with probability $E(t) = 1 - e^{-ct}$, $c > 0$. Denote as γ and θ the internal call volume and service time, respectively. Let $\Phi(x,t) = \mathbf{P}\{\gamma < x, \theta < t\}$ be the joint distribution function of random variables γ and θ. An internal call appearance interrupts external call service. After internal call arriving its service begins immediately. Let $R(x) = \Phi(x, \infty)$ and $H(t) = \Phi(\infty, t)$ be the distribution functions ofrandom variables γ and θ, respectively. After completing the internal call service (at epoch T), the interrupted service of the external call will be continued. If this service is not completed during time t, an internal demand appears in the interval $[T; T+t)$ with probability $E(t)$ and so on.

Denote as $\sigma(t)$ the total volume of calls present in the system at time moment t. Suppose that values of $\sigma(t)$ process is not limited for all $t \geq 0$. If T is an epoch of external or internal call appearance, then $\sigma(T) = \sigma(T^-) + x$, where x is the volume of the arriving call. Denote as $\eta(t)$ the number of external calls present in the system at time moment t. At the epoch t of external call appearance we have $\eta(t) = \eta(t^-) + 1$. If external call service completes at the epoch t, then $\eta(t) = \eta(t^-) - 1$.

We shall analyze two following schemes of system behavior at the epoch T of service termination.

Scheme 1. If T is the epoch of external call service termination, then $\sigma(T) = = \sigma(T^-) - x$, where x is the external call volume. If T is the epoch of internal call service termination, then $\sigma(T) = \sigma(T^-) - y$, where y is the internal call volume.

Scheme 2. If T is the epoch of internal call service termination, then $\sigma(T) = = \sigma(T^-)$. If T is the epoch of external call service termination, then $\sigma(T) = \sigma(T^-) - y$, where y is the total volume of the external call and all internal calls arriving during its service.

We shall use the following notations.

Let

$$\alpha(s,q) = \int_0^\infty \int_0^\infty e^{-sx-qt} dF(x,t)$$

and

$$\psi(s,q) = \int_0^\infty \int_0^\infty e^{-sx-qt} d\Phi(x,t)$$

be the Laplace-Stieltjes transforms (LST) of $F(x,t)$ and $\Phi(x,t)$ distribution functions respectively. Denote as

$$\alpha_{ij} = (-1)^{i+j} \frac{\partial^{i+j}\alpha(s,q)}{\partial s^i \partial q^j}\bigg|_{s=0,q=0}, \ \psi_{ij} = (-1)^{i+j} \frac{\partial^{i+j}\psi(s,q)}{\partial s^i \partial q^j}\bigg|_{s=0,q=0}$$

the mixed $(i+j)$th moments of distribution functions $F(x,t)$ and $\Phi(x,t)$, respectively, $i,j = 1,2,\ldots$. Let $\varphi(s) = \alpha(s,0)$, $\beta(q) = \alpha(0,q)$, $r(s) = \psi(s,0)$, $h(q) = \psi(0,q)$ be LST of random variables ζ, ξ, γ and θ, respectively. Denote as φ_i, β_i, r_i, h_i the ith moments of these random variables. Let $D(x,t) = \mathbf{P}\{\sigma(t) < x\}$ be the distribution function of the total volume of external and internal calls present in the system at time moment t. It is known [1] that the stability condition for the system under consideration has the form $\rho = a\beta_1 G'(1)(1 + ch_1) < 1$. If this condition takes place, then the limit $D(x) = \lim\limits_{t\to\infty} D(x,t) = \mathbf{P}\{\sigma < x\}$ exists, where σ is the stationary total calls volume.

Let $\bar{\delta}(s,t) = \displaystyle\int_0^\infty e^{-sx} d_x F(x,t)$ be the LST of distribution function $D(x,t)$ with respect to x. The Laplace transform of this function with respect to t is denoted as $\delta(s,q) = \displaystyle\int_0^\infty e^{-qt}\bar{\delta}(s,t)dt$. Our main purpose is to determine the function $\delta(s,q)$, from which all characteristics of the process $\sigma(t)$ can be determined. For example, stationary characteristics can be obtained from the function

$$\delta(s) = \int_0^\infty e^{-sx} dD(x) = \lim_{t\to\infty} \bar{\delta}(s,t) = \lim_{q\to 0} q\delta(s,q).$$

2. Random process and characteristics

First we introduce the following notation. Let $\nu(t)$ be the function taking two values: $\nu(t) = 0$, if the service of an external call takes place at the moment t, and $\nu(t) = 1$, if the service of an internal call takes place at the moment t (this function is undefined at the moment t, if there are no calls in the system at this moment). Suppose that external or internal call service takes place at the moment t. Let $\xi^*_{(0)}(t)$ be the total time of external demand service from the beginning to the moment t, if $\nu(t) = 0$ or $\nu(t) = 1$ at this moment, and $\xi^*_{(1)}(t)$ be the time from the beginning og internal call service to the moment t, if $\nu(t) = 1$ (it is clear that the function $\xi^*_{(1)}(t)$ is undefined, if $\nu(t) = 0$). Then the Markov process

$$\big(\eta(t), \nu(t), \xi^*_{(0)}(t), \xi^*_{(1)}(t)\big) \tag{1}$$

describes the system behavior. We shall characterize this process by the functions having the following probability sense:

$$P_0(t) = \mathbf{P}\{\eta(t) = 0\}; \tag{2}$$

$$P_k(0,x,t)dx = \mathbf{P}\{\eta(t) = k, \nu(t) = 0, \xi^*_{(0)}(t) \in [x; x+dx)\},$$
$$k = 1, 2, \dots ; \tag{3}$$

$$P_k(1,x,y,t)dx\,dy =$$
$$= \mathbf{P}\{\eta(t) = k, \nu(t) = 1, \xi^*_{(0)}(t) \in [x; x+dx), \xi^*_{(1)}(t) \in [y; y+dy)\},$$
$$k = 1, 2, \dots . \tag{4}$$

Let us assume for simplicity that densities $b_{(0)}(t)$ and $b_{(1)}(t)$ of random variables ξ and θ exist. Note that all results of the paper can be obtained without this assumption. Suppose that $P_0(0) = 0$ (the system is empty when $t = 0$, this is identified as zero initial condition). Denote as $\mu_{(0)}(t) = \dfrac{b_{(0)}(t)}{1 - B(t)}$ and $\mu_{(1)}(t) = \dfrac{b_{(1)}(t)}{1 - H(t)}$ the rate of an external and internal call service, respectively. If the above stability condition takes place, the following limits exist:

$$p_0 = \mathbf{P}\{\eta = 0\} = \lim_{t\to\infty} P_0(t); \tag{5}$$

$$p_k(0,x)dx = \mathbf{P}\{\eta = k, \nu = 0, \xi^*_{(0)} \in [x; x+dx)\} =$$
$$= \lim_{t\to\infty} P_k(0,x,t)dx,\ k = 1,2,\dots; \tag{6}$$

$$p_k(1,x,y)dx\,dy = \mathbf{P}\{\eta = k, \nu = 1, \xi^*_{(0)} \in [x; x{+}dx), \xi^*_{(1)} \in [y; y{+}dy)\} =$$
$$= \lim_{t\to\infty} P_k(1,x,y,t)dx\,dy,\ k = 1,2,\dots, \tag{7}$$

where η, $\xi^*_{(0)}$, $\xi^*_{(1)}$ are the stationary analogues of functions $\eta(t)$, $\xi^*_{(0)}(t)$ and $\xi^*_{(1)}(t)$, respectively.

3. Equations for the system characteristics and their solution

Let $\delta_{i,k}$ be Kronecker's symbol: $\delta_{i,j} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$

Using the method of auxiliary variables [2], we can write out partial differential equations for the functions defined by (2)–(4):

$$\frac{\partial P_0(t)}{\partial t} = -aP_0(t) + \int_0^t P_1(0,x,t)\mu_{(0)}(x)dx; \tag{8}$$

$$\frac{\partial P_k(0,x,t)}{\partial t} + \frac{\partial P_k(0,x,t)}{\partial x} = -(a + c + \mu_{(0)}(x))P_k(0,x,t) +$$
$$+ \int_0^t P_k(1,x,y,t)\mu_{(1)}(y)dy +$$
$$+(1-\delta_{k,1})a\sum_{i=1}^{k-1} P_i(0,x,t)g_{k-i}, \quad k = 1,2,\dots; \tag{9}$$

$$P_k(0,0^+,t) = \int_0^t P_{k+1}(0,x,t)\mu_{(0)}(x)dx + ag_kP_0(t),$$
$$k = 1,2,\dots; \tag{10}$$

$$\frac{\partial P_k(1,x,y,t)}{\partial t} + \frac{\partial P_k(1,x,y,t)}{\partial y} = -(a+\mu_{(1)}(y))P_k(1,x,y,t)+$$

$$+(1-\delta_{k,1})a\sum_{i=1}^{k-1} P_i(1,x,y,t)g_{k-i}, \quad k=1,2,\dots; \tag{11}$$

$$P_k(1,x,0^+,t) = cP_k(0,x,t). \tag{12}$$

Denote as

$$p_0(q) = \int_0^\infty e^{-qt}P_0(t)dt,\ p_k(0,x,q) = \int_0^\infty e^{-qt}P_k(0,x,t)dt,$$

$$p_k(1,x,y,q) = \int_0^\infty e^{-qt}P_k(1,x,y,t)dt,\ k=1,2,\dots,$$

the Laplace transforms with respext to t of the functions $P_0(t)$, $P_k(0,x,t)$ and $P_k(1,x,y,t)$, respectively.

Using the Laplace transform in equation (8) and taking into account the initial condition we obtain

$$qp_0(q) - 1 = -ap_0(q) + \int_{t=0}^\infty e^{-qt}\left(\int_0^t P_1(0,x,t)\mu_{(0)}(x)dx\right)dt,$$

where

$$\int_{t=0}^\infty e^{-qt}\left(\int_0^t P_1(0,x,t)\mu_{(0)}(x)dx\right)dt =$$

$$= \int_{x=0}^\infty \left(\int_{t=x}^\infty e^{-qt}P_1(0,x,t)dt\right)\mu_{(0)}(x)dx.$$

Using the notation $p_1^*(0,x,q) = \int_x^\infty e^{-qt}P_1(0,x,t)dt$ we obtain the following equation:

$$(q+a)p_0(q) = \int_0^\infty p_1^*(0,x,q)\mu_{(0)}(x)dx + 1. \tag{13}$$

Let us introduce the following generation functions:

$$P_{(0)}(z,x,t) = \sum_{k=1}^\infty P_k(0,x,t)z^k,$$

$$P_{(1)}(z,x,y,t)=\sum_{k=1}^{\infty}P_k(1,x,y,t)z^k,$$

$$G(z)=\sum_{k=1}^{\infty}g_k z^k.$$

Then from equations (9) we obtain the equation for the function $P_{(0)}(z,x,t)$:

$$\frac{\partial P_{(0)}(z,x,t)}{\partial t}+\frac{\partial P_{(0)}(z,x,t)}{\partial x}=-(a+c+\mu_{(0)}(x))P_{(0)}(z,x,t)+$$

$$+\int_0^t P_{(1)}(z,x,y,t)\mu_{(1)}(y)dy+a\sum_{k=2}^{\infty}z^k\sum_{i=1}^{k-1}P_i(0,x,t)g_{k-i}.$$

For the sum in the last formula we have

$$a\sum_{k=2}^{\infty}z^k\sum_{i=1}^{k-1}P_i(0,x,t)g_{k-i}=$$

$$=\sum_{i=1}^{\infty}P_i(0,x,t)z^i\sum_{k=i+1}^{\infty}z^{k-i}g_{k-i}=aG(z)P_{(0)}(z,x,t).$$

Then we obtain the following equation:

$$\frac{\partial P_{(0)}(z,x,t)}{\partial t}+\frac{\partial P_{(0)}(z,x,t)}{\partial x}=-(a+c+\mu_{(0)}(x)-aG(z))P_{(0)}(z,x,t)+$$

$$+\int_0^t P_{(1)}(z,x,y,t)\mu_{(1)}(y)dy. \tag{14}$$

Let us introduce the following Laplace transforms:

$$p_{(0)}(z,x,q)=\int_0^{\infty}e^{-qt}P_{(0)}(z,x,t)dt,$$

$$p_{(1)}(z,x,y,q)=\int_0^{\infty}e^{-qt}P_{(1)}(z,x,y,t)dt.$$

Then passing to Laplace transform in Eq. (14) we have

$$qp_{(0)}(z,x,q)+\frac{\partial p_{(0)}(z,x,q)}{\partial x}=-(c+a-aG(z)+\mu_{(0)}(x))p_{(0)}(z,x,q)+$$

$$+\int_0^\infty e^{-qt}\left(\int_0^t P_{(1)}(z,x,y,t)\mu_{(1)}(y)dy\right)dt,$$

where

$$\int_0^\infty e^{-qt}\left(\int_0^t P_{(1)}(z,x,y,t)\mu_{(1)}(y)dy\right)dt =$$

$$= \int_0^\infty p_{(1)}(z,x,y,q)\mu_{(1)}(y)dy,$$

i.e. we obtain the following equation:

$$\frac{\partial p_{(0)}(z,x,q)}{\partial x} = -(q+c+a-aG(z)+\mu_{(0)}(x))p_{(0)}(z,x,q)+$$

$$+\int_0^\infty p_{(1)}(z,x,y,q)\mu_{(1)}(y)dy. \tag{15}$$

Passing to generation functions in Eq. (10) we have

$$P_{(0)}(z,0,t) = \frac{1}{z}\int_0^t P_{(0)}(z,x,t)\mu_{(0)}(x)dx-$$

$$-\int_0^t P_1(0,x,t)\mu_{(0)}(x)dx + aG(z)P_0(t),$$

whence we obtain, passing to Laplace transforms

$$p_{(0)}(z,0,q) = \frac{1}{z}\int_0^\infty p_{(0)}(z,x,q)\mu_{(0)}(x)dx-$$

$$-\int_0^\infty p_1^*(0,x,q)\mu_{(0)}(x)dx + aG(z)p_0(q).$$

But from Eq. (13) we have

$$\int_0^\infty p_1^*(0,x,q)\mu_{(0)}(x)dx = (q+a)p_0(q)-1.$$

Then we obtain the equation

$$p_{(0)}(z,0,q) = \frac{1}{z}\int_0^\infty p_{(0)}(z,x,q)\mu_{(0)}(x)dx + 1 - (q+a-aG(z))p_0(q). \tag{16}$$

Passing to generation functions in Eq. (11) we have

$$\frac{\partial P_{(1)}(z,x,y,t)}{\partial t}+\frac{\partial P_{(1)}(z,x,y,t)}{\partial y}=-(a-aG(z)+\mu_{(1)}(y))P_{(1)}(z,x,y,t).$$

In terms of Laplace transforms we obtain the following equation

$$\frac{\partial p_{(1)}(z,x,y,q)}{\partial y}=-(q+a-aG(z)+\mu_{(1)}(y))p_{(1)}(z,x,y,q). \tag{17}$$

Passing to generation functions in Eq. (12) we have

$$P_{(1)}(z,x,0,t)=cP_{(0)}(z,x,t),$$

or (in the terms of Laplace transforms)

$$p_{(1)}(z,x,0,q)=cp_{(0)}(z,x,q). \tag{18}$$

The solution of Eq. (17) has the form (taking into account the form of the function $\mu_{(1)}(y)$)

$$p_{(1)}(z,x,y,q)=[1-H(y)]e^{-(q+a-aG(z))y}p_{(1)}(z,x,0,q)$$

or, as it follows from Eq. (18),

$$p_{(1)}(z,x,y,q)=c[1-H(y)]e^{-(q+a-aG(z))y}p_{(0)}(z,x,q). \tag{19}$$

Denote as $\pi(q)$ the PLS of busy period of the system under consideration. It is known [2] that $p_0(q)=(q+a-a\pi(q))^{-1}$. The function $\pi(q)$ will be determined later. Now we substitute relation (19) into Eq. (15) taking into account the form of the function $\mu_1(y)$. So, we obtain

$$\frac{\partial p_{(0)}(z,x,q)}{\partial x}=-(q+c-ch(q+a-aG(z))+$$
$$+a-aG(z)+\mu_{(0)}(x))p_{(0)}(z,x,q)$$

or, if we introduce the notation $\chi(z,q)=q+c-ch(q+a-aG(z))+a-aG(z)$,

$$\frac{\partial p_{(0)}(z,x,q)}{\partial x}=-(\chi(z,q)+\mu_{(0)}(x))p_{(0)}(z,x,q).$$

The solution of the last equation has the following form:

$$p_0(z,x,q)=[1-B(x)]e^{-\chi(z,q)x}p_0(z,0,q). \tag{20}$$

Then from Eq. (16) we obtain

$$p_0(z,0,q) = \frac{z[1-(q+a-aG(z))]p_{(0)}(q)}{z-\beta(\chi(z,q))},$$

and from relation (20) we have

$$p_0(z,x,q) = \frac{z[1-(q+a-aG(z))]p_{(0)}(q)}{z-\beta(\chi(z,q))}[1-B(x)]e^{-\chi(z,q)x}. \quad (21)$$

From relation (19) we obtain

$$p_{(1)}(z,x,y,q) = \frac{cz[1-(q+a-aG(z))]p_{(0)}(q)}{z-\beta(\chi(z,q))}[1-B(x)][1-H(y)]\times$$

$$\times \exp[-\chi(z,q)x-(q+a-aG(z))y]. \quad (22)$$

Let us determine the function $\pi(q)$. Let $\omega(q)$ be the LST of the random time τ from the beginning to the termination of an arbitrary call service. Then we have (as it follows from the theory of usual $M/G/1/\infty$ queue with bulk arrivals [4])

$$\pi(q) = G(\omega(q+a-a\pi(q))). \quad (23)$$

Let us determine the function $\omega(q)$. Let $\omega(q|\xi=u)$ be the conditional LST of the random variable τ under condition that the service time of a call is equal to u. It is obvious that

$$\omega(q|\xi=u) = e^{-qu}\sum_{k=0}^{\infty}\frac{(cu)^k}{k!}e^{-cu}(h(q))^k = e^{-(q+c-ch(q))u},$$

whence it follows that

$$\omega(q) = \int_0^\infty \omega(q|\xi=u)dB(u) =$$

$$= \int_0^\infty e^{-(q+c-ch(q))u}dB(u) = \beta(q+c-ch(q)),$$

i.e.

$$\omega(q+a-a\pi(q)) = \beta(q+a-a\pi(q)+c-ch(q+a-a\pi(q))),$$

and Eq. (23) takes the form

$$\pi(q) = G(\beta(q + a - a\pi(q) + c - ch(q + a - a\pi(q)))).$$

From the last relation we can determine the moments of the busy period. For example, for the first moment we have (if $\rho = a\beta_1 G'(1)(1+ch_1) < 1$)

$$\pi_1 = -\pi'(q)|_{q=0} = \frac{G'(1)\beta_1(1 + ch_1)}{1 - a\beta_1 G'(1)(1 + ch_1)}.$$

4. The non-stationary and stationary characteristics of the total calls volume

Scheme 1. For the scheme 1 we have obviously

$$D(x,t) = \mathbf{P}\{\sigma(t) < x\} = P_0(t)+$$

$$+\sum_{k=1}^{\infty}\int_0^t \mathbf{P}\{\sigma(t) < x \,|\, \eta(t) = k, \nu(t) = 0, \xi^*_{(0)}(t) = y\}P_k(0,y,t)dy+$$

$$+\sum_{k=1}^{\infty}\int\limits_{y=0}^{t}\int\limits_{u=0}^{t}\mathbf{P}\{\sigma(t) < x \,|\, \eta(t) = k, \nu(t) = 1, \xi^*_{(0)}(t) = y, \xi^*_{(1)}(t) = u\}\times$$

$$\times P_k(1,y,u,t)dy\,du. \tag{24}$$

Denote as $A * B(x)$ the Stieltjes convolution of the distribution functions $A(x)$ and $B(x)$ of non-negative random variables, i.e. $A * B(x) = \int_0^x A(x-u)dB(u)$. Denote as $A_*^{(n)}(x)$ the n-order Stieltjes convolution of the distribution function $A(x)$, $n = 0, 1, \ldots$, i.e.

$$A_*^{(0)}(x) \equiv 1,\ A_*^{(n)}(x) = \int_0^x A_*^{(n-1)}(x-u)dA(u),\ n = 1, 2, \ldots.$$

Then we have obviously

$$\mathbf{P}\{\sigma(t) < x \,|\, \eta(t) = k, \nu(t) = 0, \xi^*_{(0)}(t) = y\} = L_*^{(k-1)} * E_y^{(0)}(x),$$

where, as it follows from [3], $E_y^{(0)}(x) = \dfrac{L(x) - F(x,y)}{1 - B(y)}$.

Analogously, we obtain

$$\mathbf{P}\{\sigma(t) < x \,|\, \eta(t) = k, \nu(t) = 1, \xi^*_{(0)}(t) = y, \xi^*_{(1)}(t) = u\} =$$

$$= L_*^{(k-1)} * E_y^{(0)} * E_u^{(1)}(x),$$

where $E_u^{(1)}(x) = \dfrac{R(x) - \Phi(x,u)}{1 - H(u)}$.

Then we have from relation (24)

$$D(x,t) = P_0(t) + \sum_{k=1}^{\infty} \int_0^t L_*^{(k-1)} * E_y^{(0)}(x) P_k(0,y,t)dy +$$

$$+ \sum_{k=1}^{\infty} \int_{y=0}^t \int_{u=0}^t L_*^{(k-1)} * E_y^{(0)} * E_u^{(1)}(x) P_k(1,y,u,t)dy\,du.$$

Passing to LST of the function $D(x,t)$ with respect to x we obtain

$$\overline{\delta}(s,t) = P_0(t) + \sum_{k=1}^{\infty} \int_0^t (\varphi(s))^{k-1} e_y^{(0)}(s) P_k(0,y,t)dy +$$

$$+ \sum_{k=1}^{\infty} \int_{y=0}^t \int_{u=0}^t (\varphi(s))^{k-1} e_y^{(0)}(s) e_u^{(1)}(s) P_k(1,y,u,t)dy\,du, \tag{25}$$

where [3]

$$e_y^{(0)}(s) = [1 - B(y)]^{-1} \int_{x=0}^{\infty} e^{-sx} \int_{w=y}^{\infty} dF(x,w),$$

$$e_u^{(1)}(s) = [1 - H(u)]^{-1} \int_{x=0}^{\infty} e^{-sx} \int_{w=u}^{\infty} d\Phi(x,w).$$

Passing in (25) to Laplace transform with respect to t we obtain the following relation for the function $\delta(s,q)$:

$$\delta(s,q) = \int_0^{\infty} e^{-qt} \overline{\delta}(s,t)dt =$$

$$= p_0(q) + \sum_{k=1}^{\infty} \int_0^{\infty} e^{-qt} \left(\int_0^t (\varphi(s))^{k-1} e_y^{(0)}(s) P_k(0,y,t) dy \right) dt +$$
$$+ \sum_{k=1}^{\infty} \int_0^{\infty} e^{-qt} \left(\int_{y=0}^t \int_{u=0}^t (\varphi(s))^{k-1} e_y^{(0)}(s) e_u^{(1)}(s) P_k(1,y,u,t) dy\, du \right) dt. \tag{26}$$

It can be easy shown that

$$S_1 = \sum_{k=1}^{\infty} \int_0^{\infty} e^{-qt} \left(\int_0^t (\varphi(s))^{k-1} e_y^{(0)}(s) P_k(0,y,t) dy \right) dt =$$
$$= (\varphi(s))^{-1} \int_0^{\infty} e_y^{(0)}(s) \left(\int_y^{\infty} P_{(0)}(\varphi(s),y,t) e^{-qt} dt \right) dy =$$
$$= (\varphi(s))^{-1} \int_0^{\infty} p_{(0)}(\varphi(s),y,q) e_y^{(0)}(s) dy.$$

Then, as it follows from relation (21),

$$S_1 = \frac{1 - (q + a - aG(\varphi(s))) p_{(0)}(q)}{\varphi(s) - \beta(\chi(\varphi(s),q))} \times$$
$$\times \int_0^{\infty} \left(\int_{x=0}^{\infty} e^{-sx} \int_{w=y}^{\infty} dF(x,w) \right) e^{-\chi(\varphi(s),q)y} dy.$$

For the integral in the last relation we have

$$\int_0^{\infty} \left(\int_{x=0}^{\infty} e^{-sx} \int_{w=y}^{\infty} dF(x,w) \right) e^{-\chi(\varphi(s),q)y} dy =$$
$$= \int_{x=0}^{\infty} e^{-sx} \int_{w=0}^{\infty} dF(x,w) \int_{y=0}^{w} e^{-\chi(\varphi(s),q)y} dy =$$
$$= \frac{1}{\chi(\varphi(s),q)} \int_{x=0}^{\infty} e^{-sx} \int_{w=0}^{\infty} \left(1 - e^{-\chi(\varphi(s),q)w}\right) dF(x,w) =$$
$$= \frac{\varphi(s) - \alpha(s,\chi(\varphi(s),q))}{\chi(\varphi(s),q)}.$$

Taking into account that $p_0(q) = (q + a - a\pi(q))^{-1}$, we finally obtain

$$S_1 = \frac{a[G(\varphi(s)) - \pi(q)][\varphi(s) - \alpha(s,\chi(\varphi(s),q))]}{\chi(\varphi(s),q)[q + a - a\pi(q)][\varphi(s) - \beta(\chi(\varphi(s),q))]}.$$

In a similar way we obtain that

$$S_2=\sum_{k=1}^{\infty}\int\limits_{y=0}^{t}\int\limits_{u=0}^{t}\mathbf{P}\{\sigma(t)<x\,|\,\eta(t)=k,\nu(t)=1,\xi^*_{(0)}(t)=y,\xi^*_{(1)}(t)=u\}\times$$

$$\times P_k(1,y,u,t)dy\,du=\frac{ac[G(\varphi(s))-\pi(q)]}{\chi(\varphi(s),q))[q+a-a\pi(q)]}\times$$

$$\times\frac{[\varphi(s)-\alpha(s,\chi(\varphi(s),q))][r(s)-\psi(s,q+a-aG(\varphi(s)))]}{[\varphi(s)-\beta(\chi(\varphi(s),q))][q+a-aG(\varphi(s)))]}.$$

From relation (26) after some calculations we obtain

$$\delta(s,q)=[q+a-a\pi(q)]^{-1}\Bigg\{1+$$

$$+\frac{a[\pi(q)-G(\varphi(s))][\varphi(s)-\alpha(s,\chi(\varphi(s),q))]}{\chi(\varphi(s),q)[\beta(\chi(\varphi(s),q))-\varphi(s)]}\times$$

$$\times\left[1+\frac{c(r(s)-\psi(s,q+a-aG(\varphi(s))))}{q+a-aG(\varphi(s))}\right]\Bigg\}. \tag{27}$$

Now let us suppose that the stability condition takes place. Then for the function $\delta(s)$ after some calculations we have

$$\delta(s)=\lim_{q\to 0}q\delta(s,q)=$$

$$=p_0\Bigg\{1+\frac{\varphi(s)-\alpha(s,\kappa(\varphi(s)))}{\kappa(\varphi(s))}\times$$

$$\times\frac{a(1-G(\varphi(s)))+c(r(s)-\psi(s,a-aG(\varphi(s))))}{\beta(\kappa(\varphi(s)))-\varphi(s)}\Bigg\}, \tag{28}$$

where $p_0=1-a\beta_1G'(1)(1+ch_1)$, $\kappa(z)=c-ch(a-aG(z))+a-aG(z)$.

Note that if calls appear according to the ordinary stationary Poisson process with the rate a (for the system $M/G/1/\infty$), we have $G(z)=z$ and $G'(1)=1$ in relations (27), (28).

We can calculate arbitrary order stationary moments (if they exist) of total calls volume using relation (28). For example, for the first moment after rather complicated calculation we obtain

$$\delta_1 = \mathbf{E}\sigma = aG'(1)[\alpha_{1\,1}(1+ch_1)+c\beta_1\psi_{1\,1}]+$$

$$+\frac{a\varphi_1\left[a(G'(1))^2\left(\beta_2(1+ch_1)^2+c\beta_1 h_2\right)+\beta_1 G''(1)(1+ch_1)\right]}{2[1-a\beta_1 G'(1)(1+ch_1)]}.$$

Note that $G''(1)=0$ for the system $M/G/1/\infty$.

Scheme 2. Suppose that an external call service begins at the moment $t=0$. Denote as $\eta_1(t)$ the number of internal calls, which are not served at this moment, present in the system under consideration at the moment t under condition that the external demand service is not completed at this moment. Then for the scheme 2 we have

$$D(x,t)=\mathbf{P}\{\sigma(t)<x\}=P_0(t+$$

$$+\sum_{k=1}^{\infty}\int_0^t\sum_{l=0}^{\infty}\mathbf{P}\{\sigma(t)<x\,|\,\eta(t)=k,\eta_1(y)=l,\nu(t)=0,\xi^*_{(0)}(t)=y\}\times$$

$$\times\mathbf{P}\{\eta_1(y)=l\}P_k(0,y,t)dy+$$

$$+\sum_{k=1}^{\infty}\int_0^t\int_0^t\sum_{l=0}^{\infty}\mathbf{P}\{\sigma(t)<x\,|\,\eta(t)=k,\eta_1(y)=l,\nu(t)=1,$$

$$\xi^*_{(0)}(t)=y,\xi^*_{(1)}(t)=u\}\mathbf{P}\{\eta_1(y)=l\}P_k(1,y,u,t)dy\,du, \tag{29}$$

where obviously $\mathbf{P}\{\eta_1(y)=l\}=\dfrac{(cy)^l}{l!}e^{-cy}$. Then we have

$$\mathbf{P}\{\sigma(t)<x\,|\,\eta(t)=k,\eta_1(y)=l,\nu(t)=0,\xi^*_{(0)}(t)=y\}=$$

$$=L_*^{(k-1)}*E_y^{(0)}*R_*^{(l)}(x)$$

and

$$\mathbf{P}\{\sigma(t)<x\,|\,\eta(t)=k,\eta_1(y)=l,\nu(t)=1,\xi^*_{(0)}(t)=y,\xi^*_{(1)}(t)=u\}=$$

$$=L_*^{(k-1)}*E_y^{(0)}*E_u^{(1)}*R_*^{(l)}(x),$$

whence formula (29) takes the form

$$D(x,t) = P_0(t) +$$

$$+\sum_{k=1}^{\infty}\int_0^t \sum_{l=0}^{\infty} L_*^{(k-1)} * E_y^{(0)} * R_*^{(l)}(x)\frac{(cy)^l}{l!}e^{-cy}P_k(0,y,t)dy +$$

$$+\sum_{k=1}^{\infty}\int_0^t \int_0^t \sum_{l=0}^{\infty} L_*^{(k-1)} * E_y^{(0)} * E_u^{(1)} * R_*^{(l)}(x)\frac{(cy)^l}{l!}e^{-cy}P_k(1,y,u,t)dy\,du. \tag{30}$$

Passing in (30) to LST with respect to x we have after calculations

$$\overline{\delta}(s,t) = P_0(t) + \frac{1}{\varphi(s)}\int_0^t e^{-(1-r(s))cy}p_{(0)}(\varphi(s),y,t)dy +$$

$$+\frac{1}{\varphi(s)}\int_0^t e^{-(1-r(s))cy}e_y^{(0)}(s)\left(\int_0^t e_u^{(1)}(s)p_{(1)}(\varphi(s),y,u,t)du\right)dy.$$

Passing after that to Laplace transform with respect to t after some calculation we obtain (in a similar way, as it was done for scheme 1)

$$\delta(s,q) = [q + a - a\pi(q)]^{-1}\Bigg\{1 +$$

$$+\frac{a[\pi(q) - G(\varphi(s))][\varphi(s) - \alpha(s, c - cr(s) + \chi(\varphi(s),q))]}{[\beta(\chi(\varphi(s),q)) - \varphi(s)][c - cr(s) + \chi(\varphi(s),q)]}\times$$

$$\times\left[1 + \frac{c(r(s) - \psi(s, q + a - aG(\varphi(s))))}{q + a - aG(\varphi(s))}\right]\Bigg\}.$$

In this case we have for the function $\delta(s)$:

$$\delta(s) = \lim_{q\to 0} q\delta(s,q) = p_0\Bigg\{1 + \frac{\varphi(s) - \alpha(s, c - cr(s) + \kappa(\varphi(s)))}{c - cr(s) + \kappa(\varphi(s))}\times$$

$$\times\frac{a[1 - G(\varphi(s))] + c[r(s) - \psi(s, a - aG(\varphi(s)))]}{\beta(\kappa(\varphi(s))) - \varphi(s)}\Bigg\}.$$

In this case we obtain for the first stationary moment of total calls volume:

$$\delta_1 = \mathbf{E}\sigma = aG'(1)[\alpha_{1\,1}(1+ch_1)+c\beta_1\psi_{1\,1}]+$$

$$+\frac{a\varphi_1(1+ch_1)\,[a\beta_2(G'(1))^2(1+ch_1)+\beta_1 G''(1)]}{2[1-a\beta_1 G'(1)(1+ch_1)]}+$$

$$+\frac{acG'(1)}{2}\left[\beta_2 r_1(1+ch_1)+\frac{a\beta_1 G'(1)h_2\varphi_1}{1-a\beta_1 G'(1)(1+ch_1)}\right].$$

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