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SPARINGLY GLUED TOLERANCES

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Abstract

We introduce the notion of sparingly glued tolerances for lattices and then count their numbers in case of finite chains. We also estimate the density of sparingly glued tolerances among all glued tolerances on finite chains.

1. Introduction

The notion of congruence is one of the basic notions in Universal Algebra. Given an algebraic structure $\mathcal{A}$, an equivalence relation on the universum of the structure compatible with all operations of the structure is called a congruence of $\mathcal{A}$. The set of all congruences of $\mathcal{A}$ is denoted by $\mathrm{Con}(A)$ and it forms an algebraic lattice (with the inclusion as the standard order).

The notion of tolerance is a natural generalization of the notion of congruence. It was introduced by Chajda and Zelinka ([4]) and it is in the focus of current interest (see [10], [6], [12]) as an important tool. In Lattice Theory tolerances are indispensable in several constructions,(e.g. Hall-Dilworth gluings, Wroński sums [11]) or decompositions (e.g. atlas decomposition, Herrmann decomposition [9]) of lattices. A *tolerance relation* of an algebraic structure $\mathcal{A}$ is a reflexive and symmetric relation compatible with all operations of $\mathcal{A}$. All tolerances of a structure $\mathcal{A}$, ordered by inclusion, also form an algebraic lattice denoted by $\mathrm{Tol}(A)$ ([3]). Although the set of all congruences of $\mathcal{A}$ is a subset of the set of its tolerances, $\mathrm{Con}(A)$ need not be a sublattice of $\mathrm{Tol}(A)$.

Let $T \in \mathrm{Tol}(A)$ and $X \subseteq A$, $X \neq \emptyset$. If every two elements of X are in the relation T, then we call X a *preblock* of T. *Blocks* are maximal preblocks (with respect to inclusion). It is easy to observe that in the case when T is a congruence, blocks coincide with congruence classes of T, which means that they are pairwise disjoint. On the other hand, $T \in \mathrm{Tol}(A) \setminus \mathrm{Con}(A)$ iff there are two overlapping blocks of T.

In this paper we deal only with finite lattices. In this case blocks of a tolerance $T \in \mathrm{Tol}(L)$ of a finite lattice L are intervals of L ([2]). Therefore, if α is a block of T, then we use the notation $\alpha = [0_\alpha, 1_\alpha]$. It means that any tolerance $T \in \mathrm{Tol}(L)$ of a finite lattice L can be represented by the system of its blocks.

The following result from [7] gives a very efficient characterization of the collection of tolerance blocks in the case of finite lattices.

Lemma 1. *For a finite lattice L, let $\mathcal{C}$ be a collection of nonempty subsets of L. Then $\mathcal{C}$ is the set of all blocks of some tolerance of L iff $\mathcal{C}$ is of the form $\{[a_\gamma, b_\gamma] \colon \gamma \in \Gamma\}$, where $[a_\gamma, b_\gamma]$ are intervals of L and the following conditions are satisfied:*

(i) $\bigcup_{\gamma\in\Gamma}[a_\gamma, b_\gamma] = L$;
(ii) *for any $\gamma, \delta \in \Gamma$, $a_\gamma = a_\delta$ is equivalent to $b_\gamma = b_\delta$;*
(iii) *for any $\gamma, \delta \in \Gamma$, there exists a $\mu \in \Gamma$ such that $a_\mu = a_\gamma \vee a_\delta$ and $b_\mu \geq b_\gamma \vee b_\delta$.*

2. Characterisation

Let us notice that in case if L is a finite chain, the collection $\mathcal{C}$ of nonempty intervals of L is the set of all blocks of some tolerance of L iff they cover L and none of the intervals is included in any other. Thus,

Corollary 2.

(1) A collection $\mathcal{C}$ of subsets of the chain $L_n = \langle\{0, \ldots, n-1\}, \leq\rangle$ is the set of all blocks of some tolerance of L iff $\mathcal{C}$ is of the form $\{\alpha_i = [n_i, m_i] \colon i = 1, \ldots, k\}$ for some $1 \leq k \leq n-1$, where $n_1 = 0$, $m_k = n-1$ and $n_i < n_{i+1} \leq m_i + 1$ and $m_i < m_{i+1}$ for all $i = 1, \ldots, k$.
(2) A collection $\mathcal{C}$ of subsets of the chain $L_n = \langle\{0, \ldots, n-1\}, \leq\rangle$ is the set of all blocks of some congruence of L iff $\mathcal{C}$ is of the form $\{\alpha_i = [n_i, m_i] \colon i = 1, \ldots, k\}$ for some $1 \leq k \leq n-1$, where $n_1 = 0$, $m_k = n-1$ and $n_i < n_{i+1} = m_i + 1$ and $m_i < m_{i+1}$ for all $i = 1, \ldots, k$.

In [5] it is proved that the set of all blocks of a tolerance $T \in \mathrm{Tol}(L)$ of a finite lattice L with the order induced by the order of their smallest elements forms a lattice called *the factor lattice of L modulo T*, which will be denoted by L/T.

A tolerance T of a lattice L is called a *glued tolerance*, see [13], if its transitive closure is the total relation L^2. The sublattice of all glued tolerances of L will be denoted by $\mathrm{Glu}(L)$. Let us observe that the total relation L^2 is the greatest element both in $\mathrm{Glu}(L)$ and in $\mathrm{Con}(L)$, being their only common element. The smallest element of $\mathrm{Glu}(L)$ is called the *skeleton tolerance* of L, and it is denoted by $\Sigma(L)$. The factor lattice $L/\Sigma(L)$ is called the *skeleton* of L. It is easy to prove that $T \in \mathrm{Glu}(L)$ iff every two blocks α, β of T, such that $\alpha \prec \beta$ in L/T, overlap. Therefore, we can notice that

Corollary 3. *A collection $\mathcal{C}$ of subsets of the chain $L_n = \langle\{0,\dots,n-1\},\leq\rangle$ is the set of all blocks of some glued tolerance of L iff $\mathcal{C}$ is of the form $\{\alpha_i = [n_i, m_i]\colon i = 1,\dots,k\}$ for some $1 \leq k \leq n-1$, where $n_1 = 0$, $m_k = n-1$ and $n_i < n_{i+1} \leq m_i < m_{i+1}$ for all $i = 1,\dots,k$.*

For any lattice L we shall say that $T \in \mathrm{Tol}(L)$ is a *spare tolerance* if for any two different blocks α and β of T it holds $|\alpha \cap \beta| \leq 1$. It is clear that every congruence is a spare tolerance. We shall say that $T \in \mathrm{Tol}(L)$ is *sparingly glued* if it is both glued and spare. Let us denote by $\mathrm{SG}(L)$ the set of all sparingly glued tolerances of the lattice L. We can observe immediately that

Proposition 4. *A tolerance T of a lattice L is sparingly glued iff for every two blocks α, β of T, such that $\alpha \prec \beta$ in L/T, we have $|\alpha \cap \beta| = 1$.*

Since for every congruence T of L we have $|\alpha\cap\beta| = 0$ for any two different blocks α and β of T, the notion of sparingly glued tolerance seems to be a natural counterpart of the notion of congruence for glued tolerances.

By Corollary 2 and Proposition 4, we directly get the following characterization of spare and sparingly glued tolerances on finite chains:

Corollary 5.

(1) A collection $\mathcal{C}$ of subsets of the chain $L_n = \langle\{0,\dots,n-1\},\leq\rangle$ is the set of all blocks of some spare tolerance of L iff $\mathcal{C}$ is of the form $\{\alpha_i = [n_i, m_i]\colon i = 1,\dots,k\}$ for some $1 \leq k \leq n-1$, where $n_1 = 0$, $m_k = n-1$ and $n_i < n_{i+1}$ and $m_i < m_{i+1}$ and $n_{i+1} = m_i$ or $n_{i+1} = m_i + 1$ for all $i = 1,\dots,k$.

(2) A collection $\mathcal{C}$ of subsets of the chain $L_n = \langle\{0,\dots,n-1\},\leq\rangle$ is the set of all blocks of some sparingly glued tolerance of L iff $\mathcal{C}$ is of the form $\{\alpha_i = [n_i, m_i]\colon i = 1,\dots,k\}$ for some $1 \leq k \leq n-1$, where $n_1 = 0$, $m_k = n-1$ and $n_i < n_{i+1} = m_i < m_{i+1}$ for all $i = 1,\dots,k$.

Example. *In the pictures below one can see examples of a congruence (Fig.1), a spare tolerance which is not glued (Fig.2), a glued tolerance which is not spare (Fig.3) and a sparingly glued tolerance (Fig.4) for the six-element chain.*

Let $L_n = \langle\{0,\dots,n-1\},\leq\rangle$ be an n-element chain. We use the following notation: $t_n = |\mathrm{Tol}(L_n)|$, $q_n = |\mathrm{Con}(L_n)|$ and $g_n = |\mathrm{Glu}(L_n)|$ and $s_n = |\mathrm{SG}(L_n)|$.

In [1] we proved that

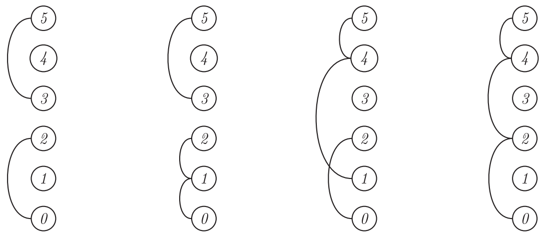

Fig.1 *Fig.2* *Fig.3* *Fig.4*

Theorem 6. *For every $n \geq 1$, we have*

(1) $g_n = c_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$;

(2) $t_n = c_n = \frac{1}{n+1}\binom{2n}{n}$,

where c_n stands for the nth Catalan number.

3. Main results

Now, our goal is to find the number of all congruences and all sparingly glued tolerances on the n-element chain. It is clear that for $n = 1$ we have $q_1 = s_n = 1$. Moreover,

Theorem 7. *(1)* $q_n = 2^{n-1}$ *for every* $n \geq 1$,

(2) $s_n = 2^{n-2}$ *for every* $n > 1$ *and* $s_1 = 1$.

Proof. (1) For $n = 1$, i.e., for the trivial lattice there is only one congruence. Then $q_1 = 1$.

Let us consider the $(n+1)$-element chain and let T be a congruence on it. The $(n+1)$st element of the cahin can constitute a single block of T - there are q_n such congruences, or it can be at the same block as the nth element of the chain - there is again q_n such congruences. Thus, we get the simple recurrence:

$$q_1 = 1; \quad q_{n+1} = 2q_n,$$

hence $q_n = 2^{n-1}$ for $n \geq 1$.

(2) It is clear that $s_1 = 1$.

Let $n > 1$ and let us consider the chain $\mathcal{C}_{n-1} = \{[i, i+1]\}_{i=0,\ldots,n-1}$ of segments of L_n with the standard order. Then the function $f : \mathcal{C}_{n-1} \mapsto L_{n-1}$ given by $f([i, i+1]) = i$ is an isomorphism.

Let T be a sparingly glued tolerance on L_n. Then every segment $[i, i+1] \in \mathcal{C}_{n-1}$ belongs to exactly one block of T. Moreover, for every block $[n_i, m_i]$ of T we have

$$f([n_i, m_i]) = f(\bigcup_{n_i \leq k \leq m_i - 1} [k, k+1]) = \bigcup_{n_i \leq k \leq m_i - 1} f([k, k+1]) = [n_1, m_i - 1].$$

Then, according to Corollaries 2 and 5, we conclude that every sparingly glued tolerance on L_n corresponds uniquely to a congruence on L_{n-1}. Therefore, according to (1), for every $n > 1$ we get

$$s_n = q_{n-1} = 2^{n-2}.$$

□

Now, let us estimate the *asymptotic density* of congruences among all tolerances of a chain and of sparingly glued tolerances among all glued tolerances of a chain. What we want to compute are the limits of sequences $({}^{q_n}/{}_{t_n})_{n\geq 1}$ and $({}^{s_n}/{}_{g_n})_{n\geq 1}$, values of which correspond to chances of drawing with uniform distribution a desired object, a congruence from the set of all tolerances or a sparingly glued tolerances from the set of all glued tolerances on a chain of length n.

In what follows, we adapt the following standard $\sim$ notation. Given two positive sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ we write $a_n \sim b_n$ whenever $\lim_{n\to\infty} {}^{a_n}/{}_{b_n} = 1$. We also apply the classical result (see [8, Chapter IV.1]) about the asymptotics of the sequence $(c_n)_{n\geq 0}$ of Catalan numbers:

$$c_n \sim 4^n n^{-3/2} \pi^{-\frac{1}{2}}.$$

Now, by Theorems 6 and 7, we get

$$\frac{q_n}{t_n} = \frac{q_n}{c_n} \sim \frac{2^{n-1}}{4^n n^{-3/2} \pi^{-\frac{1}{2}}} \overset{n\to\infty}{\longrightarrow} 0,$$
$$\frac{s_n}{g_n} = \frac{q_{n-1}}{t_{n-1}} \overset{n\to\infty}{\longrightarrow} 0.$$

This implies that the chances of drawing, both, a congruence from the set of all tolerances and a a sparingly glued tolerance from the set of all tolerances of a chain tend to zero as the length of a chain increases.

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