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# The equivalence relation as the set valued by some Heyting algebra.

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I. The relations of a partial order on a set are denoted by the symbol $\leq$. Let $< L, \leq >$ be a partially ordered set and $X$ be a non-empty subset of $L$. The least upper bound of $X$ in $L$ and the greatest lower bound of $X$ in $L$ are denoted by $\bigvee X$ and $\bigwedge X$, respectively. If $X$ is a two-element set, $X = \{a, b\}$, the respective bounds are denoted by $a \vee b$ and $a \wedge b$. *A lattice* is a partially ordered set $< L, \leq >$ with the property that for every pair $a, b$ of elements of $L$, the supremum $a \vee b$ and the infimum $a \wedge b$ exist. A lattice $\Im =< L, \leq >$ is called *complete*, if for each non-empty set $X \subseteq L$, the least upper bound $\bigvee X$ and the greatest lower bound $\bigwedge X$ exists.

Let a and $b$ be elements of a lattice $< L, \leq >$. An element $x \in L$ is called the *pseudocomplement of a relative to b*, if $x$ is the largest element of $L$ with the property that $a \wedge x \leq b$. This element is denoted by $a \rightarrow b$.

If the lattice $\Im$ possesses the least element (which is denoted by 0), then the element $a \rightarrow 0$ is called the *pseudocomplement* of a and is denoted by $\neg a$.

Any lattice with the least element 0 such that the operation of relative pseudocomplementation $\rightarrow$ is defined for every pair $a, b$ i.e., $a \rightarrow b$ exists for all $a, b$, is called a *Heyting algebra*. (Instead of „Heyting algebra" the term „pseudo-Boolean algebra" is also often used in the literature).

If a lattice with the above properties is complete, it is called a *complete Heyting algebra*.

It is a well-known fact (see e.g. [3]) that a complete lattice $\Im$ is a Heyting algebra if and only if, for every indexed subset $\{a_t\}_{t \in T}$ of this lattice and for every $a \in L$, the following equality holds in the lattice:

$$a \wedge \bigvee_{t \in T} a_t = \bigvee_{t \in T} (a \wedge a_t)$$

(The condition is referred to as infinite distributivity). It follows from this result that not every complete and distributive lattice is a Heyting algebra. For example, in the lattice of all closed subsets of a stright line there does not exists the pseudocomplement of the element $p$ relative to the empty set $\emptyset$, where $p$ is any point of the line.

Clearly, every finite distributive lattice is a Heyting algebra. Also every Boolean algebra is a Heyting algebra. It is also know (cf. [3]) that a Heyting algebra is a Boolean algebra iff, for every element a of this algebra, $a \vee \neg a = 1$, where 1 stands for the greatest element of the lattice.

II. Let $\mathcal{A} = (A, \leq)$ be a complete Heyting algebra.
By *a set valued by the algebra $\mathcal{A}$, shortly an $\mathcal{A}$ – set,* we shall mean any pair $(U, \delta)$ such that $U$ is a set and $\delta : U \times U \to A$ is a mapping satisfying the following conditions:

$$(1)\ \forall x, y \in U \quad \delta(x, y) = \delta(y, x),$$

$$(2)\ \forall x, y, z \in U \quad \delta(x, y) \wedge \delta(y, z) \leq \delta(x, z).$$

The intuitive sense of the definitionof on $\mathcal{A}$ – set is: for any two given elements $x, y$ of the set $U$, the element $\delta(x, y)$ of the Heyting algebra $\mathcal{A}$ defines the extent with respect to which the element $x$ is equal to $y$.

Let $\mathcal{A}$ be a finite Heyting algebra. Let us notice that every finite partition of the set $U$ induces a certain Heyting algebra (obviously, $\mathcal{A}$ is then a finite algebra, so it is complete) and a certain $\mathcal{A}$ – set. So,if the underling set of the algebra $\mathcal{A}$ consists of elements $\{a_1, a_2, \ldots, a_n\}$, then $U = \bigcup_{i \in T} U_i$, where

$$T = \{1, 2, \ldots, n\},\ U_i = \{x \in U : a_i \leq \delta(x, x)\}.$$

Conversely, if $\{U_i : i \in T\}$ is a finite partition of the set $U$, then we define the algebra $\mathcal{A}$ as a subalgebra of the algebra of all subsets of the set $T$ $(2^T, \cap, \cup)$ generated by set

$$\mathcal{T} = \{T_x : x \in U\}, \text{ where } T_x = \{i \in T : x \in U_i\}$$

Since the algebra $(2^T, \cap, \cup)$ is a distributive lattice, so $\mathcal{A}$ as its sublattice is also a distributive lattice and, as a consequence of this, is a Heyting algebra.

We define the function $\delta : U \times U \to A$ in the following way:

$$\delta(x, y) = T(x) \cap T(y).$$

It is easy to observe, that the pair $(U, \delta)$ is an $\mathcal{A}$ – set.

Let $\mathcal{B}_2$ be the two-element Boolean algebra and let $(U, \delta)$ be a set valued by the algebra $\mathcal{B}_2$. We define a relation $R$ on $U$ as follows:

$$xRy \iff \delta(x, y) = 1,$$

for $x, y \in U$.
Then the condition (1) of the definition of $\mathcal{A}$ – sets states, that $R$ is a symmetric relation, and the condition (2) states, that $R$ is transitive.

Let us observe that $R$ is on equivalence relation on the set $\tilde{U} = \{x \in U : \exists y \in U \quad \delta(x, y) = 1\}$, because for every $x \in \tilde{U}$ we have $\delta(x, x) = 1$ (on the ground of conditions (1) and (2)).

Now, let us assume, that $R$ is an equivalence relation on the set $U$. Then the pair $(U, \delta)$, where the function $\delta : U \times U \to \{0,1\}$ is defined in the following way:

$$\delta(x,y) = \begin{cases} 1 & \text{if } xRy, \\ 0 & \text{otherwise,} \end{cases}$$

is a $\mathcal{B}_2$ − set.

## REFERENCE


[1] G. Grätzer, General Lattice Theory, Akademie - Verlag, Berlin, 1978.

[2] D. Higgs, Injectivity in the topos complete Heyting algebra valued sets, Canad. J. Math., 36 (1984), pp. 550 - 568.

[3] H.Rasiowa and R. Sikorski, The Mathematics of Metamathematics, Warszawa (1970), PWN, Tom 41.