Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowa

Matematyka III, Częstochowa 1995/96

The block tableaux method in finitely many-valued logics

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Introduction

In the paper we will present a tableaux formalization of an arbitrary n-valued logic, $n \geq 2$. Our formalism constitutes a certain modification — in the style of Beth's semantic tableaux — of the formalization presented by W.A. Carnielli (see [Carnielli 1987]) for finitely many-valued first order logics. The difference concerns the method of representing sets of formulae in the process of theorem proving. Our method allows us to implement the algorithm for authomatic theorem proving in the simplest way possible.

1 Syntax of semantic tableaux

Assume that S is the set of all the formulae of the first order language defined in [Borowik 1993]. Let $\Gamma_0, \ldots, \Gamma_{n-1}$ be arbitrary finite sets of formulae. In particular, some of the sets Γ_i may be empty. By an *n–field semantic tableau*, shortly a *tableau*, we mean an ordered n–tuple of sets of formulae $(\Gamma_0, \ldots, \Gamma_{n-1})$ denoted as follows:

(a) $\Gamma_0 \mid \Gamma_1 \mid \cdots \mid \Gamma_{n-1}$.

Let $\Sigma = \Gamma_0 \vdash \cdots \vdash \Gamma_{n-1}$ and $\Pi = \Delta_0 \vdash \cdots \vdash \Delta_{n-1}$ be two arbitrary tableaux. The tableau Σ is said to be *contained* in the tableau Π iff $\Gamma_i \subseteq \Delta_i$ for every i, $0 \leq i \leq n-1$. The above fact will be denoted by $\Sigma \subseteq \Pi$. By the *composition* of the tableaux Σ and Π, denoted by $\Sigma * \Pi$, or shortly by $\Sigma\Pi$, we mean the tableau

$$\Lambda_0 \mid \cdots \mid \Lambda_{n-1}, \quad \text{where } \Lambda_i = \Gamma_i \cup \Delta_i \quad \text{for} \quad 0 \leq i \leq n-1.$$

Let $\Gamma \subseteq S$ be an arbitrary set of formulae. By $|_j\, \Gamma$ we mean a tableau $\Sigma = \Gamma_0 \mid \cdots \mid \Gamma_{n-1}$ such that

$$\Gamma_i = \begin{cases} \Gamma & \text{if} \quad i = j, \\ \emptyset & \text{if} \quad i \neq j. \end{cases}$$

In particular, $|_j\ \alpha$ is a tableau of the form $|_j\ \{\alpha\}$. From the above definitions it easily follows that an arbitrary tableau $\Sigma = \Gamma_0 \mid \cdots \mid \Gamma_{n-1}$ can be represented as the composition of n tableaux of the form $|_j\ \Gamma_j$ for $0 \leq j \leq n-1$. A tableau is said to be *atomic* if all the formulae appearing in this tableau are atomic.

Now we shall define conditions $s\#(c;j)$ and $w_q(d;j)$. Recall that $G_n = \{0,1\}^n - \{(0,0,\ldots,0)\}$. Let $\boldsymbol{x}, \boldsymbol{y} \in G_n$, $\boldsymbol{x} = (x_{i_1},\ldots,x_{i_n})$, $\boldsymbol{y} = (y_{i_1},\ldots,y_{i_n})$. We say that $\boldsymbol{x} < \boldsymbol{y}$ iff $x_{i_p} \leq y_{i_p}$ for $p = 1,2,\ldots,n$, and there exists p, $1 \leq p \leq n$, such that $x_{i_p} \neq y_{i_p}$. Now let $s : E_n^m \longrightarrow E_n$ be the interpretation of a connective σ, and let a function $q : G_n \longrightarrow E_n$ be the interpretation of an n–valued quantifier Q. The function s (resp. the function q) is said to *satisfy condition* $s(c_{i_o}, c_{i_1},\ldots,c_{i_{k-1}};j)$, (resp. $q(d_{k_1}, d_{k_2},\ldots,d_{k_n};j)$) for variables $x_{i_o}, x_{i_1},\ldots,x_{i_{k-1}}$ (and resp. for variables $y_{k_1}, y_{k_2},\ldots,y_{k_n}$) iff the value of the function s for $x_{i_t} = c_{i_t}$, $t = 0,1,\ldots,k-1$ (resp. the value of the function q for $y_{k_p} = d_{k_p}$, $p = 1,2,\ldots,n$), is j. The function s (the function q) is said to *minimally satisfy condition* $s(c_{i_o}, c_{i_1},\ldots,c_{i_{k-1}};j)$, (resp. condition $q(d_{k_1}, d_{k_2},\ldots,d_{k_n};j)$), which will be denoted by $s\#(c_{i_o}, c_{i_1},\ldots,c_{i_{k-1}};j)$ (and resp. by $w_q(d_{k_1}, d_{k_2},\ldots,d_{k_n};j)$, iff $k-1$ is the minimal index of a variable for which the condition is satisfied, i.e. iff for any $\{b_{i_o}, b_{i_1},\ldots,b_{i_{p-1}}\} \subseteq \{c_{i_o}, c_{i_1},\ldots,c_{i_{k-1}}\}$ such that $p \neq k$ the function s does not satisfy the condition $s(b_{i_o}, b_{i_1},\ldots,b_{i_{p-1}};j)$ for any subset of the variables
$x_1, x_2,\ldots,x_m$, (resp., there does not exist a sequence
$(a_{k_1}, a_{k_2},\ldots,a_{k_n}) < (d_{k_1}, d_{k_2},\ldots,d_{k_n})$ such that the condition
$q(d_{k_1}, d_{k_2},\ldots,d_{k_n};j))$ is satisfied. If we denote the sequences
$(c_{i_o}, c_{i_1},\ldots,c_{i_{k-1}})$ and $(d_{k_1}, d_{k_2},\ldots,d_{k_n})$ by $\boldsymbol{c_i}$ and $\boldsymbol{d_k}$, respectively, then the notations $s\#(c_{i_o}, c_{i_1},\ldots,c_{i_{k-1}};j)$ and $w_q(d_{k_1}, d_{k_2},\ldots,d_{k_n};j)$ can be respectively abbreviated to $s\#(\boldsymbol{c_i};j)$ and $w_q(\boldsymbol{d_k};j)$. For example, let s be a function interpreting three-valued Łukasiewicz implication in the matrix $\mathfrak{M}_{L3} = ((\{0,1,2\}, s, n), \{2\})$, where s and n are given by the following truth tables:

s	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

x	nx
0	2
1	1
2	0

The function s minimally satisfies condition $s(0;2)$ with respect to the first argument, and condition $s(2;2)$ with respect to the second one, since in the first case $s(0,x_2) = 2$ for any x_2, and in the second case $s(x_1,2) = 2$ for any x_1.

Now let σ be an arbitrary m–ary connective, and let a function $s : E_n^m \to E_n$ be the interpretation of this connective. Moreover, let Q be an arbitrary n–valued quantifier, and let a function $\rho : \mathbf{D}_n \to E_n$ be the interpretation of this quantifier. An $(m+1)$–tuple $(i, j_1, \ldots, j_m)$, $m < n$, is said to satisfy the condition $w_Q(i; j_1, \ldots, j_m)$, shortly $w_Q(i; \boldsymbol{j})$, if the n–tuple $(d(0), \ldots, d(j_1), \ldots, d(j_2), \ldots, d(j_m), \ldots, d(n-1))$ defined by the formula

$$d(t) = \begin{cases} 1 & \text{if } t \in \{j_1, \ldots, j_m\}, \\ 0 & \text{otherwise} \end{cases}$$

belongs to the set $\rho^{-1}(\{i\})$. In other words, the condition $w_Q(i, j_1, \ldots, j_n)$ is satisfied iff an n–tuple with 1's on the positions $j_1, \ldots, j_m$ and 0's on the remaining positions is assigned the value i by the function ρ.

Now we shall give the *elimination rules* for eliminating quantifiers Q from formulae of the type $Q_{x_k}\beta(x_k)$ and connectives σ from formulae of the type $\sigma(\alpha_1, \alpha_2, \ldots, \alpha_m)$ occuring in the j-th field of a tableau

$$\Sigma = \Gamma_0 \mid \Gamma_1 \mid \cdots \Gamma_j, \mid \ldots \mid \Gamma_{n-1}.$$

The said rules, denoted by (σj) and (Q_j), have the following forms:

$$(\sigma j) \qquad \frac{\Gamma_0 \mid \Gamma_1 \mid \cdots \mid \Gamma_j, \sigma(\alpha_1, \alpha_2, \ldots, \alpha_m) \mid \ldots \mid \Gamma_{n-1}}{\{\Sigma_{\boldsymbol{c}} : \boldsymbol{c} \in E_n^k \ \&\ k \le m \ \&\ s\#(\boldsymbol{c}; j)\}}$$

$$(Qj) \qquad \frac{\Gamma_0 \vdash \cdots \vdash \Gamma_j, Q_{x_k}\beta(x_k) \vdash \cdots \vdash \Gamma_{n-1}}{\{\Delta_{\boldsymbol{a}} : \boldsymbol{a} \in G_n \ \&\ w_q(\boldsymbol{a}; j)\}}$$

for $j = 0, 1, \ldots, n-1$,
where

$$\Sigma_{\mathbf{c}} = \Gamma_0' \mid \Gamma_1{}' \mid \cdots \mid \Gamma'_{n-1},$$

$$\Delta_{\boldsymbol{a}} = \Gamma''_0 \mid \cdots \mid \Gamma''_{n-1},$$

with

$$\Gamma_i{}' = \Gamma_i \cup \{\alpha_t : pr_t(\mathbf{c}) = i\},$$

$$\Gamma''_i = \Gamma_i \cup \{\beta(a) : (d(j_0), \ldots, d(j_{n-1})) \in \rho^{-1}(\{i\})\},$$

for $i = 0, 1, \ldots, n-1$, where $pr_k(\boldsymbol{c})$ denotes the projection of the sequence $\boldsymbol{c}$ on the k–th coordinate,

$$d(j_k) = \begin{cases} 1 & \text{if there is a sequence } b \in U^N \text{ such that} \\ & v(\beta(x/a_s), b) = k, \\ 0 & \text{otherwise.} \end{cases}$$

Here the notations $s\#(\boldsymbol{c}; j)$ and $w_q(\boldsymbol{a}; j)$ mean that the functions minimally satisfy conditions $(\boldsymbol{c}; j)$ and $(\boldsymbol{a}; j)$, respectively, with certain limitations to be imposed on the constants a_s depending on the quantifier Q eliminated by means of the considered rule.

The tableau $\Sigma = \Gamma_0 \mid \Gamma_1 \mid \cdots \mid \Gamma_j, \sigma(\alpha_1, \alpha_2, \ldots, \alpha_m) \mid \cdots \mid \Gamma_{n-1}$, or respectively the tableau $\Sigma = \Gamma_0 \mid \Gamma_1 \mid \cdots \mid \Gamma_j, Qx_k\beta(x_k) \mid \ldots \mid \Gamma_{n-1}$, is *the premise* of rule (σj), respectively (Qj), and the set $\Sigma_{\mathbf{c}}$ (resp. $\Delta_{\boldsymbol{a}}$) of tableaux — its *conclusion.* The rules $(\sigma j), (Qj)$ can be applied to a tableau Σ if it is of the form given above. In particular, all the sets Γ_i, $i = 0, 1, \ldots, n-1$, may be empty.

Now let us consider some examples. Let $\sim$ and $\Rightarrow$ denote n-valued negation and implication in Łukasiewicz logic, respectively, with the following interpretations in $\mathbf{E}_n$:

$$x \Rightarrow y = \begin{cases} n-1 & \text{if } x \leq y, \\ n-1-x+y & \text{if } x > y, \end{cases}$$

$$\sim x = n-1-x.$$

The elimination rules for the above Łukasiewicz connectives are of the following forms

$$(tln)_j \quad \frac{\Gamma_0 \mid \cdots \mid \Gamma_{j-1} \mid \Gamma_j, \sim\alpha \mid \Gamma_{j+1} \mid \cdots \mid \Gamma_{n-1}}{\Gamma_0 \mid \cdots \mid \Gamma_{n-1-j}, \alpha \mid \cdots \mid \Gamma_{n-1}},$$

$$(tli)_j \quad \frac{\Gamma_0 \mid \cdots \mid \Gamma_{j-1} \mid \Gamma_j, \alpha \Rightarrow \beta \mid \Gamma_{j+1} \mid \cdots \mid \Gamma_{n-1}}{\Gamma_0 \mid \cdots \mid \Gamma_s, \alpha \mid \cdots \mid \Gamma_t, \beta \mid \cdots \mid \Gamma_{n-1} : i\#(s;j), i\#(s,t;j), i\#(t;j)\}},$$

where

$$i(s,t) = \begin{cases} n-1 & \text{if } i \leq t, \\ n-1-s+t & \text{if } i > t, \end{cases} \quad \text{for any } j \in E_n.$$

For Słupecki's logic with the primary connectives $\Rightarrow, \sim, \neg$, interpreted in Słupecki's matrices by the functions

$$x \Rightarrow y = \begin{cases} n-1 & \text{if } 0 \leq x < r, \\ y & \text{if } r \leq x \leq n-1, \end{cases}$$

$$\sim x = x -_n 1 = \begin{cases} x-1 & \text{if } 0 < x \leq n-1, \\ n-1 & \text{if } x = 0, \end{cases}$$

$$\neg x = \begin{cases} x & \text{if } 0 \leq x \leq n-3, \\ n-2 & \text{if } x = n-1, \\ n-1 & \text{if } x = n-2, \end{cases}$$

the rules for elimination of the above-mentioned connectives are of the following forms:

$$(tsi)_j \quad \frac{\Gamma_0 \mid \cdots \mid \Gamma_j, \alpha \Rightarrow \beta \mid \cdots \mid \Gamma_{n-1}}{\{\Gamma_0 \mid \cdots \mid \Gamma_s, \alpha \mid \cdots \mid \Gamma_t, \beta \mid \cdots \mid \Gamma_{n-1} : i\#(s;j), i\#(s,t;j), i\#(t;j)\}},$$

$$\text{where } i(s,t) = \begin{cases} n-1 & \text{if } 0 \leq s < r, \\ t & \text{if } r \leq s \leq n-1, \end{cases}$$

$$(tsn)_0 \quad \frac{\Gamma_0, \sim\alpha \mid \Gamma_1 \mid \cdots \mid \Gamma_{n-1}}{\Gamma_0 \mid \Gamma_1 \mid \cdots \mid \Gamma_{n-1}, \alpha},$$

$$(tsn)_j \quad \frac{\Gamma_0, \mid \cdots \mid \Gamma_{j-1} \mid \Gamma_j, \sim\alpha \mid \Gamma_{j+1} \mid \cdots \mid \Gamma_{n-1}}{\Gamma_0 \mid \cdots \mid \Gamma_{j-1}, \alpha \mid \Gamma_j \mid \Gamma_{j+1} \mid \cdots \mid \Gamma_{n-1}}, \quad 0 < j \leq n-1,$$

$$(tsm)_j \quad \frac{\Gamma_0, \mid \Gamma_1 \mid \cdots \mid \Gamma_j, \neg\alpha \mid \cdots \Gamma_{n-1}}{\Gamma_0, \alpha \mid \Gamma_1 \mid \cdots \mid \Gamma_j, \alpha \mid \cdots \Gamma_{n-1}, \alpha}, \qquad 0 \leq j \leq n-3,$$

$$(tsm)_{n-2} \quad \frac{\Gamma_0, \mid \Gamma_1 \mid \cdots \mid \Gamma_{n-2}, \neg\alpha \mid \Gamma_{n-1}}{\Gamma_0 \mid \Gamma_1 \mid \cdots \mid \Gamma_{n-1}, \alpha},$$

$$(tsm)_{n-1} \quad \frac{\Gamma_0, \mid \Gamma_1 \mid \cdots \mid \Gamma_{n-1}, \neg\alpha}{\Gamma_0 \mid \Gamma_1 \mid \cdots \mid \Gamma_{n-2}, \alpha \mid \Gamma_{n-1}}.$$

In Sobociński's logic, where the implication $\Rightarrow$ i negation $\sim$ are respectively interpreted by the functions

$$x \Rightarrow y = \begin{cases} n-1 & \text{if } x = y, \\ y & \text{if } x \neq y, \end{cases}$$

$$\sim x = \begin{cases} 0 & \text{if } x = n-1, \\ x+1 & \text{if } 0 \leq x < n-1, \end{cases}$$

the rules for elimination of these connectives take the following forms:

$$(tsn)_j \quad \frac{\Gamma_0 \mid \cdots \mid \Gamma_j, \sim\alpha \mid \cdots \mid \Gamma_{n-1}}{\Gamma_0 \mid \cdots \mid \Gamma_{j+1}, \alpha \mid \cdots \Gamma_{n-1}}, \quad j \in E_n, j \neq n-1,$$

$$(tsn)_{n-1} \quad \frac{\Gamma_0 \mid \Gamma_1 \mid \cdots \mid \Gamma_k, \mid \cdots \mid \Gamma_{n-1}, \sim\alpha}{\Gamma_0, \alpha \mid \Gamma_1 \mid \cdots \mid \Gamma_k \mid \cdots \mid \Gamma_{n-1}},$$

$$(tsi)_j \quad \frac{\Gamma_0 \mid \cdots \mid \Gamma_j, \alpha \Rightarrow \beta \mid \cdots \mid \Gamma_{n-1}}{\{\Gamma_0 \mid \cdots \mid \Gamma_s, \alpha \mid \cdots \mid \Gamma_t, \beta \mid \cdots \mid \Gamma_{n-1} : s \neq t, s \Rightarrow t = j\}},$$
$$j \neq n-1,$$

$$(tsi)_{n-1} \quad \frac{\Gamma_0 \mid \Gamma_1 \mid \cdots \mid \Gamma_{n-1}, \alpha \Rightarrow \beta}{\begin{array}{c}\{\Gamma_0 \mid \cdots \mid \Gamma_s, \alpha, \beta \mid \cdots \mid \Gamma_{n-1} : s \in E_n\}; \\ \{\Gamma_0 \mid \cdots \mid \Gamma_s, \alpha \mid \cdots \mid \Gamma_{n-1}, \alpha : s < n-1\}\end{array}}$$

For implicatively-negational n-valued propositional Post logics, in which implication and negation are respectively interpreted by the functions

$$x \Rightarrow y = \begin{cases} n-1, & \text{if } x \le y, \\ n-1-x+y, & \text{if } x > y \text{ and } x < r, \\ y, & \text{if } x > y \text{ and } x \ge r, \end{cases}$$

$$\sim x = \begin{cases} n-1 & \text{if } x = 0, \\ x-1 & \text{if } x \neq 0, \end{cases}$$

the rules for elimination of the above connectives are of the following forms:

$$(tpi)_j \quad \frac{\Gamma_0 \mid \cdots \mid \Gamma_j, \alpha \Rightarrow \beta \mid \cdots \mid \Gamma_{n-1}}{\{\Gamma_0 \mid \cdots \mid \Gamma_s, \alpha \mid \cdots \mid \Gamma_t, \beta \mid \cdots \mid \Gamma_{n-1} : i\#(s;j), i\#(s,t;j), i\#(t;j)\}}$$

$$\text{where } i(s,t) = \begin{cases} n-1, & \text{if } s \le t, \\ n-1-s+t, & \text{if } s > t \text{ and } s < r, \\ t, & \text{if } s > t \text{ and } s \ge r, \end{cases}$$

$$(tpn)_o \quad \frac{\Gamma_0, \sim\alpha \mid \Gamma_1 \mid \cdots \mid \Gamma_{n-1}}{\Gamma_0 \mid \Gamma_1 \mid \cdots \mid \Gamma_{n-1}, \alpha},$$

$$(tpn)_j \quad \frac{\Gamma_0 \mid \cdots \mid \Gamma_{j-1} \mid \Gamma_j, \sim\alpha\ \Gamma_{j+1} \mid \cdots \mid \Gamma_{n-1}}{\Gamma_0 \mid \cdots \mid \Gamma_{j-1}, \alpha \mid \Gamma_j \mid \Gamma_{j+1} \mid \cdots \mid \Gamma_{n-1}}, \quad 0 < j \le n-1.$$

The three-valued propositional logic introduced by Bočvar and Finn [Bočvar, Finn 1976] is determined by the matrix

$$\mathfrak{M}_{BF_3} = (E_3, \{2\}, \sim, \wedge, \vee, \Rightarrow, \Leftrightarrow),$$

where the functions $\sim, \wedge, \vee, \Rightarrow, \Leftrightarrow$ are defined as follows:

x	$\sim x$
0	2
1	1
2	0

$\wedge$	0	1	2
0	0	1	0
1	1	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

$\Rightarrow$	0	1	2
0	2	2	2
1	2	2	2
2	0	0	2

$\Leftrightarrow$	0	1	2
0	2	2	0
1	2	2	0
2	0	0	2

Of course, $\alpha \Leftrightarrow \beta$ is an abbreviation for $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$. Both the function $\Leftrightarrow$ and the corresponding connective will be disregarded in the following considerations.

The rules for eliminating the connectives $\sim, \wedge, \vee, \Rightarrow$ from the formulae occuring in a tableau have the following schemata:

$$(tbfn)_0 \quad \frac{\Gamma, \sim\alpha \mid \Pi}{\Gamma \mid \Delta \mid \Pi, \alpha},$$

$$(tbfn)_1 \quad \frac{\Gamma \mid \Delta, \sim\alpha \mid \Pi}{\Gamma \mid \Delta, \alpha \mid \Pi},$$

$$(tbfn)_2 \quad \frac{\Gamma \mid \Delta \mid \Pi, \sim\alpha}{\Gamma, \alpha \mid \Delta \mid \Pi},$$

$$(tbfk)_0 \quad \frac{\Gamma, \alpha \wedge \beta \mid \Delta \mid \Pi}{\Gamma, \alpha, \beta \mid \Delta \mid \Pi;\ \Gamma, \alpha \mid \Delta \mid \Pi, \beta;\ \Gamma, \beta \mid \Delta \mid \Pi, \alpha},$$

$$(tbfk)_1 \quad \frac{\Gamma \mid \Delta, \alpha \wedge \beta \mid \Pi}{\Gamma \mid \Delta, \alpha \mid \Pi;\ \Gamma \mid \Delta, \beta \mid \Pi},$$

$$(tbfk)_2 \quad \frac{\Gamma \mid \Delta \mid \Pi, \alpha \wedge \beta}{\Gamma \mid \Delta \mid \Pi, \alpha, \beta},$$

$$(tbfa)_0 \quad \frac{\Gamma, \alpha \vee \beta \mid \Delta \mid \Pi}{\Gamma, \alpha, \beta \mid \Delta \mid \Pi},$$

$$(tbfa)_1 \quad \frac{\Gamma \mid \Delta, \alpha \vee \beta \mid \Pi}{\Gamma \mid \Delta, \alpha \mid \Pi;\ \Gamma \mid \Delta, \beta \mid \Pi},$$

$$(tbfa)_2 \quad \frac{\Gamma \mid \Delta \mid \Pi, \alpha \vee \beta}{\Gamma, \alpha \mid \Delta \mid \Pi, \beta;\ \Gamma, \beta \mid \Delta \mid \Pi, \alpha;\ \Gamma \mid \Delta \mid \Pi, \alpha, \beta},$$

$$(tbfi)_0 \quad \frac{\Gamma, \alpha \Rightarrow \beta \mid \Delta \mid \Pi}{\Gamma, \beta \mid \Delta \mid \Pi, \alpha;\ \Gamma \mid \Delta, \beta \mid \Pi, \alpha},$$

$$(tbfi)_2 \quad \frac{\Gamma \mid \Delta \mid \Pi, \alpha \Rightarrow \beta}{\Gamma, \alpha \mid \Delta \mid \Pi;\ \Gamma \mid \Delta, \alpha \mid \Pi;\ \Gamma \mid \Delta \mid \Pi, \beta}.$$

In the sequel it will turn out that the tableau $\Gamma \mid \Delta, \alpha \Rightarrow \beta \mid \Pi$ is inconsistent, and the connective $\Rightarrow$ need not be eliminated from the formula $\alpha \Rightarrow \beta$.

Quantifiers for n-valued first-order logic — denote them by $Q_j, j \in W$, where W is a certain set of indices — can be interpreted as functions

$$q_j : \mathcal{P}(E_n) - \emptyset \to E_n.$$

In particular, the existential quantifier $\exists$ is interpreted by the function *max*, and the universal quantifier $\forall$ — by *min*. The rules for eliminating the quantifiers $\exists$ and $\forall$ are respectively of the following forms:

$$(\exists j)_n \quad \frac{\Sigma * \mid_j \exists x \alpha(x)}{\Sigma * \mid_j \alpha(a), \exists x \alpha(x)} \quad \text{for } 0 \leq j < r,$$

$$(\exists j)_w \quad \frac{\Sigma * \mid_j \exists x \alpha(x)}{\Sigma * \mid_j \alpha(b)} \quad \text{for } r \leq j \leq n-1,$$

$$(\forall j)_n \quad \frac{\Sigma * \mid_j \forall x \alpha(x)}{\Sigma * \mid_j \alpha(c)} \quad \text{for } 0 \leq j < r,$$

$$(\forall j)_w \quad \frac{\Sigma * \mid_j \forall x \alpha(x)}{\Sigma * \mid_j \alpha(d), \forall_v x \alpha(x)} \quad \text{for } r \leq j \leq n-1.$$

On the constants a, b, c, d, we impose certain conditions depending on the type of the quantifier Q eliminated, as well as on the table field index j and on the number n of logical values. In case of Bočvar-Finn's [Bočvar, Finn 1976] first order three-valued predicate calculus, the rules for the existential quantifier $\exists$ and the universal quantifier $\forall$ have respectively the following schemata:

$$(tbf\exists)_o \quad \frac{\Gamma, \exists_x\alpha(x) \mid \Delta \mid \Pi}{\Gamma\alpha(a), \exists_x\alpha(x) \mid \Delta \mid \Pi},$$

where a is an arbitrary element of some universe U,

$$(tbf\exists)_1 \quad \frac{\Gamma \mid \Delta, \exists_x\alpha(x) \mid \Pi}{\Gamma \mid \Delta, \alpha(b) \mid \Pi},$$

$$(tbf\exists)_2 \quad \frac{\Gamma \mid \Delta \mid \Pi, \exists_x\alpha(x)}{\Gamma \mid \Delta \mid \Pi, \alpha(b), \alpha, \alpha(a), \exists_x\alpha(x);\ \Gamma, \alpha(a), \exists_x\alpha(x) \mid \Delta \mid \Pi},$$

where a is an arbitrary constant, and b does not occur in any tableau on the branch B of the proof tree which contains the tableau with the formula $\exists_x\alpha(x)$,

$$(tbf\forall)_o \quad \frac{\Gamma, \forall_x\alpha(x) \mid \Delta \mid \Pi}{\Gamma, \alpha(b), \alpha(a), \forall_x\alpha(x) \mid \Delta \mid \Pi;\ \Gamma \mid \Delta \mid \Pi, \alpha(a), \forall_x\alpha(x)}$$

with the limitations on b like in $(\exists_2)$,

$$(tbf\forall)_1 \quad \frac{\Gamma \mid \Delta, \forall_x\alpha(x) \mid \Delta}{\Gamma \mid \Delta, \alpha(b) \mid \Delta},$$

with the limitations on b like in $(\forall_o)$,

$$(tbf\forall)_2 \quad \frac{\Gamma \mid \Delta \mid \Pi, \forall_x\alpha(x)}{\Gamma \mid \Delta \mid \Pi, \alpha(a), \delta\forall_x\alpha(x)},$$

where a is an arbitrary element of a given universe U.

A tableau $\Sigma = \Gamma_0 \mid \cdots \mid \Gamma_{n-1}$ is said to be *inconsistent* iff

(a) there exist r, s such that $0 \leq r \leq n-1, 0 \leq s \leq n-1$ and $\Gamma_r \cap \Gamma_s \neq \emptyset$, or

(b) there exists a non-atomic formula $\sigma(\alpha_1, \ldots, \alpha_m) \in \Gamma_j$, for which there does not exist a sequence of constants $\boldsymbol{c} \in E$ such that $s\#(\boldsymbol{c}, j)$, or

(c) there exists a formula of the form $Q_x\alpha(x) \in \Gamma_j$, for which there does not exists a sequence of constants $\boldsymbol{a} \in \boldsymbol{G_n}$ such that $w_q(\boldsymbol{a}; j)$.

A tableau Σ is said to be *atomic* if all the sets Γ_j on its fields $(j = 0, 1, \ldots, n-1)$ contain at most atomic formulae.

Recall that S denotes the set of all well-formed formulae of the language of first order calculus of n–valued predicates. Let U be an arbitrary nonempty set of individuals. The set U is called a *universe of individuals,* shortly *universe* or *domain.* The set S_U of U*formulae* is defined analogously as the set S; the only difference is that every occurence of any symbol representing an individual constant in a formula is replaced by a suitable element of the universe U. Hence by an *atomic U–formula* we mean an expression of the form $p_i(z_1, z_2, \ldots z_m)$, where p_i is an m–ary predicate symbol, and z_i is either an individual variable or an element of the universe U for $1 \leq i \leq m$. Hence S_U can be defined inductively as follows:

$$S_{U,0} = \{\alpha : \alpha \text{ is an atomic } U\text{–formula }\},$$

$$S_{U,k+1} = S_{U,k} \cup \{\sigma_i(\alpha_1, \alpha_2, \ldots, \alpha_m) : \alpha_1, \ldots, \alpha_m \in S_{U,k}, \sigma_i \in \mathbf{A}\} \cup$$

$$\cup \{Q_w x_i \alpha : \alpha \in S_{U,k}, Q_w \in \mathbf{Q}, x_i \in \mathbf{V}\}, \quad \text{and}$$

$$S_U = \bigcup_{n \in N} S_{U,n}$$

2. Remarks about semantics of n–valued predicate calculus

In propositional calculus, the values of formulae are determined by the values of their subformulae. However, as we have noticed above, quantified formulae may have infinitely many subformulae. In order to establish the relationship between the truth values of quantified formulae and the values of their subformulae, we employ an additional function, sometimes called the *distribution function.*

Let $n > 0$ and $m > 0$, and let $E_n = \{0, \ldots, n-1\}, s_j : E_n^m \longrightarrow E_n$ for $j \in L, q_w : \mathcal{P}(E_n) \backslash \{0\} \longrightarrow E_n$ for $w \in W$, where $\mathcal{P}(E_n)$ denotes the powerset of the set E_n. Let $\mathbf{D}_n = \mathcal{P}(E_n) - \{0\}$. One can easily note that there exists a natural one-to-one correspondence between the set $\mathbf{D}_n$ and the set $G_n = \{0,1\}^n - \{(0, 0, \ldots, 0)\}$, and hence these sets will be sometimes identified.

A quadruple

$$\mathcal{E} = (E_n, E_n^*, \{s_j : j \in J\}, \{q_w : w \in W\}),$$

where $E_n^* \subseteq E_n, E_n^* \neq E_n$, is called a *structure* for the set S. In addition, we assume that $E_r^* = \{r, r+1, \ldots n-1\}$ and $r > 0$. Now let U be and arbitrary nonempty set, and let $\mathfrak{R}$ denote the following system:

$$\mathfrak{R} = (U, \{p_i : i \in I\}, \{g_j : j \in J\}, \{a_k : k \in K\}, \mathcal{E}),$$

where p_i and g_j are functions such that

$$p_i : U^{arg(p_i)} \longrightarrow E \ \text{ for } \ i \in I,$$

$$g_j : U^{arg(g_j)} \longrightarrow E \ \text{ for } \ j \in J, \quad \text{and}$$

$$a_k \in U \ \text{ for } k \in K,$$

i.e. the a_k's are certain selected elements of the set U. (In the above relationships, $arg(p_i)$ denotes arity of the function p_i, and the meaning of $arg(g_j)$ is analogous).

The notions of interpretation of the language S_U in the structure $\mathfrak{R}$, as well as of satisfiability and truth of formulae are defined in a standard way.

For technical reasons, we shall now define a certain special valuation, called a Post valuation. Let S_U^c denote the set of all closed U–formulae of the first order language. By a *Post valuation* we mean a valuation $v : S_U^c \longrightarrow E_U$ such that the structure $\mathcal{E}$ is a chain-complete Post algebra of order U. By an *atomic valuation* we mean a function

$$v_A : At_U \to E_U,$$

where At_U denotes the set of all atomic U–formulae without free variables.

Obviously, if two atomic valuations v_A and w_A coincide on the set At_U, then they coincide also on the whole set S_U^c. Hence an arbitrary atomic valuation of the set At_U can be extended to a valuation on S_U^c in at most one way. One can also easily note that an interpretation in a structure $\mathfrak{R}$ is closely connected with an atomic valuation. Any given atomic valuation v_A generates some corresponding interpretation in a certain structure $\mathcal{E}_{v_A}$, and conversely: an interpretation i generates a corresponding atomic valuation v_{i_A}.

3. Proof tree of a tableau.

By a *proof tree* of a tableau

$$\Sigma = \Gamma_0 \mid \Gamma_1 \mid \cdots \mid \Gamma_{n-1}$$

we mean a quintuple

$$\mathcal{D} = (\Sigma, D, D' \, r, l),$$

where:

(a) the tableau Σ is the root of the tree,

(b) D is a set of tableaux, and $D' \subseteq D$,

(c) r is a binary relation on D, defined as follows:

$\Delta r \Pi \Leftrightarrow$ there exists a rule (σj), with $\sigma \in \mathbf{A}$, $j \in E_n$, such that the tableau Δ is the premise of this rule, and the tableau Π – one of the elements of its conclusion; moreover, each vertex of the tree except the root has exactly one predecessor,

(d) $l : D \to N$ is a function satisfying the following conditions:

$l(\Sigma) = 0$,

if $\Delta r \Pi$, then $l(\Pi) = l(\Delta) + 1$.

The value $l(\Delta)$ of the function l for Δ is called *the level of a tableau* Δ in the tree $\mathcal{D}$. The set D' is a set of final tableaux, also referred to as leaves of the tree $\mathcal{D}$. By a *final tableau* we mean one that is either

(a) an inconsistent tableau , or

(b) an atomic tableau.

The tableaux that fulfil condition (a) above are referred to as *closed tableaux.* In the opposite case, a final tableau is said to be *open.* A branch of the tree is said to be *closed* or *inconsistent* if it contains a closed tableau; otherwise it is said to be *open.* A proof tree of a tableau Σ is said to be *closed* if all its branches are closed. Recall that r, where $0 < r \leq n-1$, is the least distinguished value in the set E_n of logical values. We say that a formula $\alpha \in S$ is a *theorem* in the semantic tableau system for n–valued logical calculus iff proof trees of the following tableaux are closed:

$$|_0\ \alpha,\ |_1\ \alpha,\ \ldots,\ |_{r-1}\ \alpha.$$

The relation r can be extended to a relation r_p which orders the vertices of the tree as follows: if Δ, Π are arbitrary vertices of the tree $\mathcal{D}$, then $(\Delta, \Pi) \in r_p$ iff either $\Delta = \Pi$, or $(\Delta, \Pi) \in r$, or there exists a sequence $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$, $m > 2$, of vertices such that $\Sigma_1 = \Delta$, $\sigma_m = \Pi$ and $(\Sigma_i, \Sigma_{i+1}) \in r$ for $i = 1, \ldots, m$. Hence the pair $(\mathcal{D}, r_p)$ is an ordered set. Each maximal chain of the tree $(\mathcal{D}, r_p)$ is called a branch of the tree $\mathcal{D}$.

Note that each of the rules we have described above, which eliminate propositional connectives and quantifiers from the formulae occuring in a tableau, represents a two-level tree. The root of that tree (level zero) is a tableau being the premise of the rule. The sequence of its k conclusions, $1 \leq k \leq m^n$ (where n is the number of logical values, and m denotes the arity of the connective eliminated), constitutes the first level of the tree. Each tableau being a conclusion of a given rule is an immediate successor of the tableau being the premise of the rule. This is illustrated by the following picture:

A proof tree $\mathcal{D}_2$ is said to be a *simple extension* of a proof tree $\mathcal{D}_1$ iff the tree $\mathcal{D}_2$ is obtained from the tree $\mathcal{D}_1$ by applying the rules (σ_j) or (Q_j) for $j \in E_n$ in the following way:

(a) if we apply a rule (σ_j) to a tableau $\Sigma \in \mathcal{D}_1$, then to each leaf Δ of the branch G of the tree $\mathcal{D}_1$ which contains Σ we attach as immediate succesors of Δ all the conclusions of the rule (σ_j) applied to the tableau Σ,

(b) if we apply a rule (Q_j) to a tableau $\Sigma \in \mathcal{D}_1$, then, similarly as in case (a), to each leaf Δ of the branch G of the tree $\mathcal{D}_1$ containing Σ we attach as immediate successors of Δ all conclusions of the rule (Q_j) applied to the tableau Σ, respecting the limitations on the constants introduced in the conclusions; moreover, if the quantifier Q is eliminated from a formula $Qx\beta(x)$, and $a_1, a_2, \ldots a_s$ is a sequence of all constants that occur on the branch G containing Σ, then on a level $t, 1 \leq t \leq s$, we extend each branch passing through Σ with successors according to the following schema:

$$\bigvee_{i=0}^{t} \{|_{k_i}\ \beta(x//a_t) : w_q(k_1, k_2, \ldots, k_n; j)\}.$$

$\mathcal{D}_\alpha$ is said to be a *proof tree of the formula* $\alpha \in S$, shortly a *tree of* α, iff there exists a sequence of trees $\mathcal{D}_1, \mathcal{D}_2, \ldots \mathcal{D}_k$ such that $\mathcal{D}_k = \mathcal{D}_\alpha, \mathcal{D}_1$ is a singleton tree consisting of the formula α, and, for each $s, 1 \leq s < k$, the tree $\mathcal{D}_{s+1}$ is a simple extension of the tree $\mathcal{D}_s$.

Let $X \subseteq S$, and define

$$\mathcal{T}^*(X) = \{|_j\ \alpha : j \in E_r^*, \alpha \in X\},$$

$$\mathcal{T}^n(X) = \{|_k\ \alpha : k \in E_n - E_r^*, \alpha \in X\},$$

$$\mathcal{T}(X) = \mathcal{T}^*(X) \cup \mathcal{T}^n(X).$$

If X is a singleton consisting of the formula α, i.e. $X = \{\alpha\}$, then the sets $\mathcal{T}^*(X), \mathcal{T}^n(X)$ and $\mathcal{T}(X)$ will be denoted by $\mathcal{T}^*\alpha, \mathcal{T}^n\alpha$ and $\mathcal{T}\alpha$, respectively. Hence $\mathcal{T}\alpha = \{|_o\ \alpha,\ |_1\ \alpha,\ \ldots,\ |_{n-1}\ \alpha\}$.

Let Y be an arbitrary subset of the set $\mathcal{T}(X)$ for some $X \subseteq S$. By a *configuration* of the set Y of tableaux we mean a proof tree $\mathcal{D}_Y$ constructed inductively in the following way:

(a) We order the elements of Y into a sequence $|_{i_1}\ \alpha_1,\ |_{i_2}\ \alpha_2, \ldots$

(b) We construct a proof tree of the tableau $|_{i_1}\ \alpha_1$ (initial step of the induction).

(c) Assume we have created a configuration for the tableaux $|_{i_1}\ \alpha_1,\ \ldots,\ |_{i_k}\ \alpha_k$. Then:

- we terminate the construction procedure, if the configuration constructed up to now does not contain an open branch, or else
- to each leaf l of an open branch of the currently created configuration we attach a proof tree of the tableau $|_{i_{k+1}} \alpha_{k+1}$ in such a way that the root of the latter tree is an immediate successor of the leaf l.

(d) If the construction procedure was not terminated in (c), we terminate it after exhausting the elements of the set Y, if Y is a finite set. In the opposite case the configuration is an infinte tree.

One can easily note that applications of the rules $(\exists_j)_n$ and $(\forall_j)_w$ may result in an infinite open tree. In practice, if we impose no limitations on the constants introduced according to a given rule, then the rule is applied at most the number of times equal to the number of constants that have occurred on a given branch before a rule $(\exists_j)_n$ or $(\forall_j)_w$ was applied. Of course, the above practical limit of the length of a branch (where the length of a branch denotes the cardinality of the set of its elements) is applied to the trees which are attached in the procedure of constructing a configuration.

Lemma 1.
Let $Y \subseteq \mathcal{T}(X)$ be an infinite set of tableaux. If a configuration $\mathcal{D}_Y$ is open, then $\mathcal{D}_Y$ contains an infinite branch.

Proof.
Since the set Y is infinite, and the configuration is an open, finitely generated tree, then the König's lemma implies existence of an infinite branch.
■

A tableau

$$\Sigma = \Gamma_0 \mid \dots \mid \Gamma_{n-1}$$

is said to be satisfiable iff there exists a valuation $v : At_U \to E_n$ such that $h_v(\alpha) = j$ for some $\alpha \in \Gamma_j$, where h_v denotes extension of the valuation v to a homorphism of S_u into E_n.

A proof tree of a tableau Σ is aid to be *satisfiable* iff some vertex of this tree is a satisfiable tableau.

Theorem 1

Let $\alpha \in S$. If a tree $\mathcal{D}$ of a tableau $|_j \alpha$ is closed, then for every valuation $v : At_U \to E_n$, $h_v(\alpha) \neq j$.

Proof.
If there existed a valuation v such that $h_v(\alpha) = j$, then the tree consisting solely of the formula $|_j \alpha$ would be satisfiable. One can easily check that

the rules $(\sigma j)(Q_j)$ extend satisfiable trees to satisfiable trees. Hence the proof tree $\mathcal{D}$ of the formula $|_j\ \alpha$ would also be satisfiable, which contradicts the fact that $\mathcal{D}$ is closed. ■

Let $Y \subseteq \mathcal{T}(X)$ for some $X \subseteq S_U$. The set Y is said to be *consistent with respect to the universe* U iff the following conditions are satisfied:

(n_1) for any atomic formula $\alpha \in S_U$ and any $j, k \in E_n$, if $j \neq k$ and $|_j\ \alpha \in Y$, then $|_k\ \alpha \notin Y$,

(n_2) if $\alpha = \sigma(\alpha_1, \ldots \alpha_m)$ and $|_j\ \alpha \in Y$, then there exist a rule (σj) and indices k, k_t, $0 \leq k \leq n-1$, $1 \leq k_t \leq m$, such that $|_k\ \alpha_{k_t} \in Y$, $|_j\ \alpha$ is the premise of (σj), and $|_k\ \alpha_{k_t}$ is a conclusion of that rule. $|_j\ \alpha$ such that $|_k\ \alpha \in X$,

(n_3) if $\alpha = Q_x\beta$ and $|_j\ \alpha \in Y$, then there exists $\boldsymbol{j} = (j_1, j_2, \ldots, j_m)$, $j_s < n$ for $s = 1, 2, \ldots, m$, such that:

(a) $w_Q(\boldsymbol{j}, j)$,

(b) for every $c \in U$ there exists s, $1 \leq s \leq m$, such that $|_s\ \beta(x/c) \in Y$,

(c) for every j_s that occurs in the sequence $\boldsymbol{j}$ there exists a constant $c \in U$ such that $|_{j_s}\ \beta(x/c) \in Y$.

For example, we shall define in a more detailed way the notion of a consistent set in three-valued Bočvar-Finn's logic. The considerd example is especially interesting in that neither disjunction nor conjunction are interpreted in E_3 as the maximum and minimum functions, respectively. In order to obtain a clear description of a consistent set in this logic, let us divide the rules $(tbfn)_o - (tbf\forall)_2$ into the following groups:

$(r_A) = \{(tbfn)_o, (tbfn)_1, (tbfn)_2, (tbfk)_2, (tbfa)_o\}$,

$(r_B) = \{(tbfk)_1, (tbfa)_1, (tbfi)_o\}$,

$(r_C) = \{(tbfk)_o, (tbfa)_2, (tbfi)_2\}$,

$(r_D) = \{(tbf\exists)_o, (tbf\forall)_2\}$,

$(r_E) = \{(tbf\exists)_1, (tbf\forall)_1\}$,

$(r_F) = \{(tbf\exists)_2, (tbf\forall)_o\}$.

The tableaux which may occur as premises of of rules in groups $(r_A) - (r_F)$ wiil be denoted by $\Sigma_A, \Sigma_B, \Sigma_C, \Sigma_D, \Sigma_E, \Sigma_F$, respectively. The tableaux being respectively conclusions of these rules will be denoted by:

Σ_{A_1} or $\Sigma_{A_1A_2}$; $\Sigma_{B_{11}B_{12}}$, $\Sigma_{B_{21}B_{22}}$, $\Sigma_{C_{11}C_{12}}$, $\Sigma_{C_{21}C_{22}}\Sigma_{C_{31}C_{32}}$, $\Sigma_{D_1(b)}$; $\Sigma_{E_1(b)}, \Sigma_{F_1(a,b)}, \Sigma_{F_2(b)}$.

Let $Y \subseteq \mathcal{T}(X), X \subseteq S_U$, where S_U is the set of all well-formed U-formulae of the language of Bočvar - Finn's three-valued predicate calculus [Bočvar, Finn 1976]. The set Y is consistent iff the following conditions are satisfied:

(b_1) for any atomic $\alpha \in S$ and $j = 0, 1, 2$, the tableau $|_j\ \alpha$ is in Y iff, for $k = 0, 1, 2$, $k \neq j$ implies $|_k\ \alpha \notin Y$,

(b_2) for any $\alpha, \beta \in S_U$, the set Y contains no tableau having a sub-tableau of the form $|_1\ \alpha \Rightarrow \beta$,

(b_3) if $\Sigma_A \in Y$, then $\Sigma_{A_1} \in Y$, or correspondingly $\Sigma_{A_1A_2} \in Y$,

(b_4) if $\Sigma_B \in Y$, then either $\Sigma_{B_{11}B_{12}} \in Y$, or $\Sigma_{B_{21}B_{22}} \in Y$,

(b_5) if $\Sigma_C \in Y$, then either $\Sigma_{C_{11}C_{12}} \in Y$, or $\Sigma_{C_{21}C_{22}} \in Y$, or $\Sigma_{C_{31}C_{32}} \in Y$,

(b_6) if $\Sigma_D \in Y$, then $\Sigma_{D_1(a)} \in Y$ for any $a \in U$,

(b_7) if $\Sigma_E \in Y$, then $\Sigma_{E_1(b)} \in Y$ for some $b \in U$,

(b_8) if $\Sigma_F \in Y$, then either $\Sigma_{F_1(a,b)} \in Y$ for some a and any b in U, or $\sigma_{F_2(b)} \in Y$ for every $b \in U$.

Returning to our discussion of an arbitrary n-valued logic, we should note that each consistent set may be extended to a maximal consistent set, i.e. a set determined by the present condition (n_1), and conditions $(n_2), (n_3)$ strengthened to equivalencies. A maximal consistent set will be also called a set of true statements.

Lemma 2
Let $Y \subseteq \mathcal{T}(S_U)$ be a consistent set. Then there exists a valuation $v : At \longrightarrow E_n$ such that, for any formula $\alpha \in S_U$, $h_v(\alpha) = j \Longleftrightarrow |_j\ \alpha \in Y$.

Proof
Let X be a set, and let the conditions $(n_1), (n_2)$ i (n_3) be satisfied. We define $v : At \rightarrow E_n$ as follows:

$$v(\alpha) = \begin{cases} j \text{ if } |_j\ \alpha \in X \text{ for atomic } \alpha, \\ \text{anything in the opposite case.} \end{cases}$$

Of course, v is a function by (n_1). It is easy to show that, for any $\alpha \in S$, $h^v(\alpha) = j$ implies $|_j\ \alpha \in X$. We will prove it by induction. For atomic

α we have $h^v(\alpha) = v(\alpha) = j$, and $|_j\, p \in Y$ by definition of v. Now let $\alpha = \sigma(\alpha_1, \dots, \alpha_m)$, and assume that $h^v(\alpha_i) = j_i$ implies $mid_{j_i}\alpha_i \in Y$ for $1 \leq i \leq m$ (inductive assumption). Since $h^v(\alpha) = h^v(\sigma(\alpha_1, \dots, \alpha_m) = s(h^v(\alpha_i), \dots, h^v(\alpha_m))$, and the set Y is consistent, then $s\#(j_1, \dots, j_m; j)$ and there exista a rule (σj) such that $(|_j\, \alpha,\ |_{j_i}\, \alpha_i) \in (\sigma j)$ for $1 \leq i \leq m$. This yields $|_j\, \alpha \in Y$. If $\alpha = Qx\beta(x)$ and $|_j\, Qx\beta(x) \in Y$, then condition (n_3) implies that $w_Q(j_1, j_2, \dots, j_m; j)$ is satisfied for some $j_1, j_2, \dots, j_m$, and for every s, $1 \leq s \leq m$, there exists $a \in U$ such that $mid_{j_s}\beta(x//a) \in Y$. Then from our inductive assumption it follows that $h_v(\beta(x//a)) = j_s$. By the inductive assumption, the above condtion is satisfied for $s = 1, 2, \dots, n$. Hence by condition $w_Q(j_1, j_2, \dots, j_m; j)$ and the defintion of the valuation we have $h_v(Qx\beta(x)) = j$. ■

Let $\mathcal{Z}$ be a *finitary property,* which will be denoted shortly by (fp), i.e. a property such that a set Y has the property $\mathcal{Z}$ iff all finite subsets of Y have the property $\mathcal{Z}$. Now let $\mathcal{K} \subseteq \mathcal{T}(S_U^C)$ be a (fp), and let $Y \subseteq S_U^C$. The set Y of sentences is said to be *$\mathcal{K}$-consistent* iif there exist a $X \in \mathcal{K}$ such that $X = \mathcal{T}^*(Y)$ (recall that $\mathcal{T}^*(Y) = \{|_j\, \alpha : j \in E_r^*, \alpha \in Y\}$). A family $\mathcal{R} \subseteq \mathcal{T}(S_U^C)$ is said to *analytically consistent* iff for every $Y \in \mathcal{R}$ the following conditions are satisfied:

(s$_1$) for any atomic α, and any $i, j \in E_n$, if $i \neq j$ and $|_i\, \alpha \in Y$, then $|_j\, \alpha \notin Y$,

(s$_2$) if $|_i\, \sigma(\beta_1, \beta_2, \dots, \beta_m) \in Y$, then there exist a rule (σi) and a tableau $|_j\, \beta_k$ such that the premise of (σi) is $|_i\, \sigma(\beta_1, \beta_2, \dots, \beta_m)$, $|_j\, \beta_k$ belongs to the set of conclusions of (σi), and $Y \cup \{|_j\, \beta_k\} \in \mathcal{R}$,

(s$_3$) if $|_i\, Qx\alpha(x) \in Y$, then there exists a finite sequence $j_1, j_2, \dots, j_k$ such that condition $w_Q(j_1, j_2, \dots, j_k; i)$ is satisfied and there exists a set X with following properties:

  (*i*) for every formula $\alpha(x/a), a \in U$ there exists s, $1 \leq s \leq k$, such that $|_{j_s}\, \alpha(x/a) \in X$,

  (*ii*) for every j_s, $1 \leq s \leq k$, there exists $a \in U$ such that $|_{j_s}\, \alpha(x/a) \in X$,

  (*iii*) $Y \cup X \in \mathcal{R}$.

Lemma 3

Let $\mathcal{R}$ be an analytically consistent family, and let Y be an $\mathcal{R}$-consistent set. Then each configuration for the set Y is open.

Proof

Let Y be $\mathcal{R}$-consistent, and let $\mathcal{D}$ be a finite configuration containing a branch G such that $Y \cup G$ is $\mathcal{R}$-consistent. If we extend branch G to a branch G_1 using rules of the type $(\sigma j)(Qj)$, then the set $Y \cup G_1$ is $\mathcal{R}$-consistent by conditions (s_2) and (s_3) of the definition of analytic consistency. Hence if Y is $\mathcal{R}$-consistent, then in the construction of the configuration for Y on each level there exists at least one branch G such that $Y \cup G$ is $\mathcal{R}$-consistent; by condition (s_1), G must be open. ■

Theorem 2
Let $\mathcal{R}$ be an analytically consistent family of sets of formulae, and let Y be an $\mathcal{R}$-consistent set. Then the set Y is satisfiable in any structure with a countable universe.

Proof
Let Y be an $\mathcal{R}$-consistent set of formulae. Then there exists a set $X \in \mathcal{R}$ such that $X = \mathcal{T}^*(Y)$. Let us order the set X of tableaux into a sequence

$$\gamma_1, \gamma_2, \ldots, \gamma_k, \ldots.$$

We will create a consistent sequence of tableaux $\varphi_1, \varphi_2, \ldots, \varphi_k, \ldots$ as follows:

(a) $\varphi_1 = \gamma_1$,

(b) suppose that we have already constructed the m-th element of the sequence, $m \geq 1$. If the elements $\varphi_1, \varphi_2, \ldots, \varphi_m$ of the sequence form a set which belongs to $\mathcal{R}$, then we extend the above sequence as follows:

(*i*) if $\varphi_m = |_i\ \sigma(\alpha_1, \alpha_2, \ldots, \alpha_s)$, then we attach to the sequence $\varphi_1, \varphi_2, \ldots, \varphi_m$ the element $|_{j_t}\ \alpha_t, \gamma_{m+1}$, obtaining the sequence

$$\varphi_1, \varphi_2, \ldots, \varphi_m, |_{j_t}\ \alpha_t, \gamma_{m+1},$$

where the pair $(|_i\ \sigma(\alpha_1, \alpha_2, \ldots, \alpha_s), |_{j_t}\ \alpha_t)$ belongs to the rule (σi) and satisfies condition (s_2) of analytic consistency,

(*ii*) if the tableau φ_m is of the form $|_i\ Qx\alpha(x)$ and the sequence $j_1, j_2, \ldots j_t$ satisfies condition (s_3) of analytic consistency definition, then we extend the sequence $\varphi_1, \varphi_2,$ $\ldots, \varphi_m$ to the sequence $\varphi_1, \varphi_2, \ldots, \varphi_m,\ |_{j_1}\ \alpha(c_1), |_{j_2}\ \alpha(c_2), \ldots, |_{j_t}$ $\alpha(c_t), \gamma_{m+1}$, where some of the constants $c_1, c_2, \ldots, c_t$, (not necessarily all of them) are subject to the limitations following from applying rule (Qi).

Evidently, the set Z of the elements of the sequence $\varphi_1, \varphi_2, \ldots, \varphi_k, \ldots$ is a consistent set for the universe consisting of the constants

$$c_{1_1}, c_{1_2}, \ldots, c_{1_{t1}}, c_{2_1}, c_{2_2}, \ldots, c_{2_{t2}}, \ldots, c_{m_1}, c_{m_2}, \ldots, c_{m_{tm}}, \ldots$$

The set of the above-mentioned constants is obviously countable. By Lemma 2, the set Z is satisfiable. ■

Lemma 4.
Let Y be an arbitrary set of sentences of a language of n-valued first-order predicate calculus. If the set Y is satisfiable, then there exists an open configuration for some set $Z \subseteq \mathcal{T}^*(Y)$.

Proof
Immediate by the satisfiability of the set Y and Lemma 2. ■

A formula α is said to be a *tableau consequence* of a set X, which will be denoted shortly by $\alpha \in CtX$, iff there exists a closed configuration for each of the sets

$$\mathcal{T}^*(X) \cup \{|_j \alpha\} \text{ for } j \in E_n - E_r^*$$

(recall that E_r^* denotes the set of distinguished values).

A formula α is said to a be a *semantic consequence* of a set X, which will be shortly denoted by $\alpha \in C_{\mathfrak{M}}X$, iff for any valuation $v : At \longrightarrow E_n$ and any $\beta \in X$, $h_v(\beta) \in E_r^*$ implies $h_v(\alpha) \in E_r^*$.

Theorem 3
If $\alpha \in C_{\mathfrak{M}}X$, then $\alpha \in CtX$.

Proof
Let $\alpha \in C_{\mathfrak{M}}$. Assume that $\alpha \notin CtX$. Then there exists $j \in E_n - E_r^*$ such that the configuration $\mathcal{T}^*(X) \cup \{|_j \alpha\}$ is open. Hence the above configuration has at least one open branch G containing the tableau $|_j \alpha$. The set of elements of G is $\mathcal{R}$-consistent and satisfiable. Hence there exists a valuation $v_o : At \longrightarrow E_n$ such that $h_{v_o}(\alpha) = j$ i $j < r$. This implies $h_{v_o}(\alpha) \notin E_r^*$ ■

Corollary 1. (completeness of the block tableau system)
Let $Y \cup \{\alpha\} \subseteq S_U^C$. Then $\alpha \in CtY$ iff $\alpha \in C_{\mathfrak{M}}Y$. ■

Corollary 2.
If $Y \subseteq S_U^C$, then Y is satisfiable iff each of its finite subsets $X \subseteq Y$ is satisfiable.

Proof
Of course, if there exists an open configuration for the set $\mathcal{T}(Y)$, then there also exists an open configuration $\mathcal{T}(X)$ for some finite subset $X \subseteq Y$. By Corollary 1, this implies Corollary 2. ■

Corollary 3.
If a set $Y \subseteq S_U^C$ is satisfiable, then Y is also satisfiable in a structure with a countable universe.

Proof
Immediate from the completeness theorem and Lemma 4. ■

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