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# On a characterization of the logarithm by a mean value property

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Any real polynomial $f(x) = ax^2 + bx + c$, $x \in \mathbb{R}$, has the property that

$$\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x+y}{2}\right)$$

for every $(x, y) \in \mathbb{R}$, $x \neq y$. It turns out that that particular form of the Lagrange mean value theorem characterizes polynomials of at most second degree. Much more can be proved: J. Aczél [1] has shown that, with no regularity assumptions, a triple $(f, g, h)$ of functions mapping $\mathbb{R}$ into itself satisfies the equation

$$\frac{f(x) - g(y)}{x - y} = h(x + y)$$

for all $(x, y) \in \mathbb{R}$, $x \neq y$, if and only if there exist real constants $a, b, c$ such that $f(x) = g(x) = ax^2 + bx + c$, $x \in \mathbb{R}$, and $h(x) = ax + b$, $x \in \mathbb{R}$. Generalizations involving weighted arithmetic means were also considered (see e.g. M. Falkowitz [3] and the references therein) and characterizations of polynomials of higher degrees (in the same spirit) were obtained (see [4] and [5], for instance).

In what follows we are going to characterize the *logarithm* in a similar way. To this end, denote by $D$ the open first quadrant of the real plane $\mathbb{R}^2$ with the diagonal removed, i.e.

$$D := (0, \infty)^2 \setminus \{(x, x) \in \mathbb{R}^2 : x \in (0, \infty)\}.$$

Applying the classical Lagrange mean value theorem to the logaritmic function we derive the existence of a function
$D \ni (x, y) \longrightarrow \xi(x, y) \in \operatorname{int\,conv}\{x, y\}$ such that the equality

$$\frac{\log x - \log y}{x - y} = \frac{1}{\xi(x, y)}$$

holds true for all pairs $(x, y) \in D$. Obviously, we have

$$\xi(x,y) = \frac{x-y}{\log x - \log y} \qquad \text{for all} \qquad (x,y) \in D.$$

A natural question arises to find all differentiable functions $f : (0, \infty) \longrightarrow \mathbb{R}$ satisfying the equation

$$\frac{f(x)-f(y)}{x-y} = f'\left(\frac{x-y}{\log x - \log y}\right)$$

for every $(x, y) \in D$.

Actually, inspired by the above mentioned Aczél's result from [1], we deal with somewhat more sophisticated problem, involving a Pexider-type functional equation, in order to have no regularity properties whatsoever. Namely, we shall prove the following

**Theorem 1.** *Let functions $f, g, h$, defined on the positive half-line, satisfy the equation*

$$\frac{f(x)-g(y)}{x-y} = h\left(\frac{x-y}{\log x - \log y}\right) \tag{1}$$

*for all $x, y \in (0, \infty)$, $x \neq y$. Then there exist real constants $a, b, c$ such that*

$$f(x) = g(x) = a\log x + b\,x + c\,, \qquad x \in (0, \infty)\,,$$

*and*

$$h(x) = a\frac{1}{x} + b\,, \qquad x \in (0, \infty)\,.$$

*Conversely, each triple $(f, g, h)$ of the form described above yields a solution to equation* (1).

*Proof.* Assume that functions $f, g, h : (0, \infty) \longrightarrow \mathbb{R}$ satisfy equation (1) for all $x, y \in (0, \infty)$, $x \neq y$. Interchanging the roles of $x$ and $y$ in (1) we leave the right hand side unchanged; therefore,

$$f(x) - g(x) = g(y) - f(y), \quad x, y \in (0, \infty),\ x \neq y\,.$$

Setting $\alpha := f(1) - g(1)$, we deduce that $g(y) = f(y) + \alpha$ for all $y \in (0, \infty) \setminus \{1\}$ and, consequently,

$$f(x) - (f(x) + \alpha) = (f(y) + \alpha) - f(y) \ \text{for all}\ x, y \in (0, \infty),\ x \neq y,\ x \neq 1 \neq y,$$

whence $\alpha = 0$. Thus $g(y) = f(y)$ for all $y \in (0, \infty) \setminus \{1\}$ and in view of the definition of $\alpha$, we have $f(1) = g(1)$ as well, so that $g = f$.

Now, equation (1) assumes the form

$$f(x)-f(y)=G\left(\frac{x-y}{\log x-\log y}\right)\log\frac{x}{y}$$

for all $x,y\in(0,\infty)$, $x\neq y$, where we have put

(2) $$G(t):=th(t),\qquad t\in(0,\infty).$$

For the sake of brevity, we set additionally

$$\varphi(t):=\frac{t-1}{\log t},\qquad t\in(0,\infty)\setminus\{1\},$$

getting

$$f(x)-f(y)=G\left(y\varphi(\frac{x}{y})\right)\log\frac{x}{y},\qquad x,y\in(0,\infty),\ x\neq y.$$

Replacing here $x$ by $xy$ we arrive at

(3) $$f(xy)-f(y)=G\left(y\varphi(x)\right)\log x,\qquad x,y\in(0,\infty),\ x\neq 1,$$

whence, subsequently, for all $x,y,z\in(0,\infty)$, $xz\neq 1\neq x$, we obtain the equalities

$$f(xyz)-f(yz)=G\left(yz\varphi(x)\right)\log x$$

and

$$f(xyz)-f(xz)=G\left(y\varphi(xz)\right)\log xz.$$

This implies that

$$f(yz)-f(y)=G\left(y\varphi(xz)\right)\log xz-G\left(yz\varphi(x)\right)\log x$$

holds true for all $x,y,z\in(0,\infty)$, $xz\neq 1\neq x$. Consequently, applying (3) to the left hand side of the latter equality, we infer that

(4) $$G\left(y\varphi(z)\right)\log z=G\left(y\varphi(xz)\right)\log xz-G\left(yz\varphi(x)\right)\log x$$

is satisfied whenever $x,y,z\in(0,\infty)$, $xz\neq 1, z\neq 1\neq x$. Putting here $z=x$ and taking into account that

$$\varphi(x^2)=\frac{x+1}{2}\varphi(x)\qquad\text{for all}\quad x\in(0,\infty)\setminus\{1\},$$

we conclude that

$$G\left(y\varphi(x)\frac{x+1}{2}\right)=\frac{G\left(xy\varphi(x)\right)+G\left(y\varphi(x)\right)}{2}$$

provided that $x,y\in(0,\infty)$ and $x\neq 1$. This proves that the function $G$ is a solution to the Jensen functional equation; indeed, fixing arbitrarily a pair $(s,t)\in(0,\infty)^2$, $s\neq t$, and putting

$$x:=\frac{t}{s}\qquad\text{and}\qquad y:=\frac{s}{\varphi\left(\frac{t}{s}\right)}$$

we have $s = y\varphi(x)$ and $t = xy\varphi(x)$, whence

$$G\left(\frac{s+t}{2}\right) = \frac{G(s)+G(t)}{2} \tag{5}$$

for all $s, t \in (0,\infty)$, including $s = t$, as claimed.

It is well known (see e.g. M. Kuczma [6, p. 315]) that (5) implies the existence of an *additive* function $A : \mathbb{R} \longrightarrow \mathbb{R}$ and a constant $a \in \mathbb{R}$ such that

$$G(x) = A(x) + a \quad \text{for all} \quad x \in (0,\infty)\,.$$

Applying this representation we can rewrite (4) as

$$A\left(y(\varphi(z) - \varphi(xz))\right) \log z = A\left(y(\varphi(xz) - z\varphi(z))\right) \log x\,,$$

whenever $x, y, z \in (0,\infty)$, $xz \neq 1, z \neq 1 \neq x$. Take here $z := e$ and $x := e^t$; then $xz \neq 1 \neq x$ if and only if $t \in \mathbb{R} \setminus \{-1, 0\}$. Thus, by means of the oddness of $A$ (resulting from its additivity),

$$A\left(y(\varphi(e^{t+1}) - \varphi(e))\right) = A\left(y(e\varphi(e^t) - \varphi(e^{t+1}))\right) \cdot t\,, \tag{6}$$

holds true for all $y \in (0,\infty)$ and all $t \in \mathbb{R} \setminus \{-1, 0\}$. One can easily check that for every $t > 0$ the number $e\varphi(e^t) - \varphi(e^{t+1})$ is positive as well, which enables to put

$$y := \frac{1}{e\varphi(e^t) - \varphi(e^{t+1})}$$

in (6) provided that $t \in (0,\infty)$. Hence, a simple calculation shows that,

$$A(t) = A\left(\frac{\varphi(e^{t+1}) - \varphi(e)}{e\varphi(e^t) - \varphi(e^{t+1})}\right) = A(1) \cdot t \quad \text{for all} \quad t \in (0,\infty)\,.$$

Setting $b := A(1)$ and recalling (2) we get

$$xh(x) = G(x) = A(x) + a = bx + a \quad \text{for all} \quad x \in (0,\infty)\,,$$

i.e.

$$h(x) = a\frac{1}{x} + b \quad \text{for all} \quad x \in (0,\infty)\,.$$

Consequently, in view of (1) and the fact that $g = f$ we conclude that

$$\frac{f(x) - f(y)}{x - y} = a\,\frac{\log x - \log y}{x - y} + b \quad \text{for all} \quad x, y \in (0,\infty),\ x \neq y\,.$$

In other words,

$$f(x) - a\log x - bx = f(y) - a\log y - by$$

for every $x, y \in (0,\infty)$, including $x = y$. By setting

$$c := f(x) - a\log x - bx \equiv \text{const} \quad \text{on} \quad (0,\infty),$$

we obtain the desired form of $f$.

The latter part of the assertion is a subject for a straightforward verification which completes the proof.

As simple consequences of Theorem 1 we obtain the following characterizations of logarithms.

**Theorem 2.** *A nonzero function* $f : (0,\infty) \longrightarrow \mathbb{R}$ *is a logarithm if and only if* $f(1) = 0$ *and there exist functions* $g, h : (0,\infty) \longrightarrow \mathbb{R}$ *such that equation* (1) *is satisfied and* $h(2) = \frac{1}{2}h(1)$.

*Proof.* Assume that a nonzero function $f : (0,\infty) \longrightarrow \mathbb{R}$ with $f(1) = 0$ satisfies equation (1) with some functions $g, h : (0,\infty) \longrightarrow \mathbb{R}$ and $h(2) = \frac{1}{2}h(1)$. Then, according to Theorem 1, there exist real constants $a, b, c$ such that

$$f(x) = g(x) = a \log x + bx + c\,, \qquad x \in (0,\infty)\,,$$

and

$$h(x) = a\frac{1}{x} + b\,, \qquad x \in (0,\infty)\,.$$

Now $\frac{1}{2}(a+b) = \frac{1}{2}h(1) = h(2) = \frac{1}{2}a + b$ yielding $b = 0$ which jointly with the equality $f(1) = 0$ gives $f(x) = a \log x$, $x \in (0,\infty)$. If we had $a = 0$ we would get $f = 0$, which contradicts our assumption. Thus

$$f(x) = \log_p x, \quad x \in (0,\infty), \qquad \text{with} \qquad p := e^{\frac{1}{a}}\,.$$

To prove the converse, assume that $f(x) = \log_p x$, $x \in (0,\infty)$, with some $p \in (0,\infty) \setminus \{1\}$. Then $f(x) = \frac{1}{\log p} \log x$, $x \in (0,\infty)$, and taking $g := f$, $h(x) := \frac{1}{x \log p}$, $x \in (0,\infty)$, we obviously have $h(2) = \frac{1}{2}h(1)$, whereas the triple $(f, g, h)$ yields a solution to equation (1).

**Theorem 3.** *A nonzero function* $f : (0,\infty) \longrightarrow \mathbb{R}$ *is a logarithm if and only if* $f(1) = 0$, $f(x_o^2) = 2f(x_o)$ *for some* $x_o \in (0,\infty) \setminus \{1\}$ *and there exist functions* $g, h : (0,\infty) \longrightarrow \mathbb{R}$ *such that equation* (1) *is satisfied.*

The proof is similar.

**Remark.** The function $\xi$ spoken of at the beginning of this paper provides an example of a mean value. Indeed,

$$D \ni (x,y) \longrightarrow \xi(x,y) = \frac{x-y}{\log x - \log y} \in \operatorname{int} \operatorname{conv} \{x,y\} \text{ for all } (x,y) \in D\,,$$

and $\lim_{y\to x}\xi(x,y) = x$, for any $x \in (0,\infty)$. What makes this *logarithmic* mean value particularly interesting is that it separates the classical geometric and arithmetic means, i.e.

$$\sqrt{xy} < \frac{x-y}{\log x - \log y} < \frac{x+y}{2} \qquad \text{for all} \qquad (x,y) \in D,$$

(see B. C. Carlson [2]; a different proof, involving the Schur-concavity machinery, may be found in A. W. Marshall & I. Olkin monograph [7, pp. 98-99]).

For these reasons the idea of characterizating the logarithm with the aid of logarithmic mean value seems to be well motivated.

**Acknowledgement.** The original statement of Theorem 1 contained the assumption of the local boundedness of function $h$ occurring at the right-hand side of equation (1). I am indebted to Professor Janusz Matkowski for his information that while considering some more general problem he had obtained a similar result (with $g = f$) with no regularity assumption (cf. [8, Theorem 3]). This stimulated me to seek for an entirely elementary direct proof of Theorem 1. The method presented here is completely different and shorter than that from [8]; therefore, I believe that it may present an interest of its own.

## References

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