Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka IV, Częstochowa 1998

Quasiarithmetic mean

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This talk through the mediation of easy motive examples gives reasons for proper establishing of the quasiarithmetic mean and gives instruction to solve practical problems that common aim is to find „an average" of certain values.

Now let's start with solved problems to demonstrate saying:

Not all the averages are the same.

Problem 1 *Cyclist stood in the middle of a hill and started going up the hill with constant speed $u = 10\ km/h$. On the hill he turned and went down again with constant speed $d = 50\ km/h$ accurately as long as he went up. What mean speed did he read on his tachometer after finishing his ride?*

Problem 2 *Cyclist was going up the hill with constant speed $u = 10\ km/h$. On the hill he turned and went down again with constant speed $d = 50\ km/h$ all the way back to a place, where he started to ride. What mean speed did he read on his tachometer after finishing his ride?*

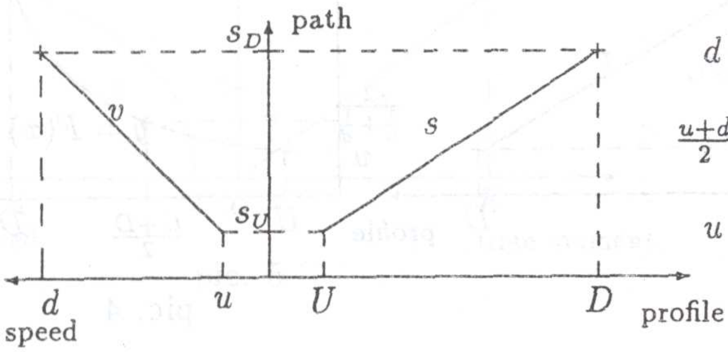
Problem 3 *Cyclist was going up the hill with constant speed $u = 10\ km/h$. At the end of his going up he started to move faster and later by the time he was going down he increased his speed for the second time. The speed was $d = 50\ km/h$. At the same time he realized that the speed on the tachometer increased as many times as it increased for the first time. What speed he was moving after the first increasing?*

Giving all these problems has something common – it gives us some information about certain two ordered pairs, for example

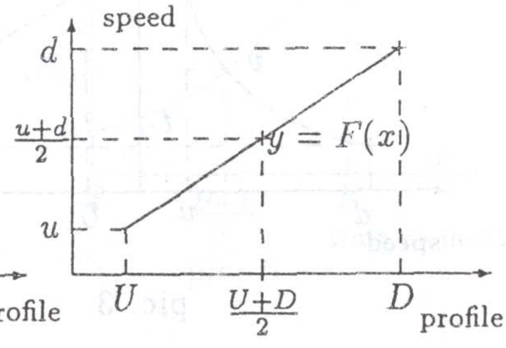
$$(U, u) \qquad \text{and} \qquad (D, d),$$

where U, resp. D indicates in first two problems the way up resp. down. (So i.e. pair (U, u) tells us that for the way up was speed u.) Let's give a graphic description of it: Let's put speed (in km/h) on perpendicular axis and the profile of the road on horizontal axis. Let's put on this axis interval $< U, D >$, which end-points represent real roads and interior-points „fictive" roads, and we know that the road continuously varies from real rising up road U, over non-exists plain road $\frac{U+D}{2}$, till real falling road D.

Now let's solve first problem. From giving the problem we see that the time of cyclist's moving is not important, because he needs for the way up and down the same time. It's also clear that as much the road is falling as faster it is and so the cyclist on the road covers through the given time longer path, it means that between the path and the profile of the road is relation of direct proportionality (on pic. 1 is demonstrated by linear function s). It's also accepted that the longer the path is (the path is covered in given time), the bigger is the speed (on pic. 1 is this demonstrated as linear function v).



pic. 1



pic. 2

Let's define function $F := v \circ s$ and draw its graph (the result of composition of two linear functions is again linear function) – see pic. 2.

Let's write function $F(x)$.

$$y = F(x) = u + \frac{d-u}{D-U}(x-u). \tag{1}$$

The task of the first problem is to find a speed, which the cyclist was moving on roads U a D, that means to find a speed, which the cyclist would move on fictive plain road $\frac{U+D}{2}$. Let's substitute for x expression $\frac{U+D}{2}$ into relation (1). So we get

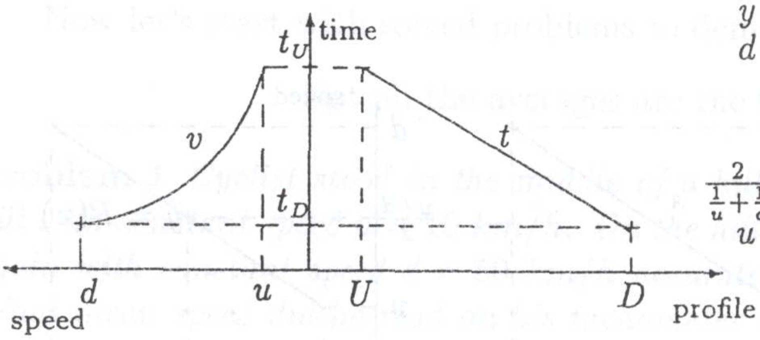
$$y = u + \frac{d-u}{D-U}\left(\frac{U+D}{2} - U\right)$$

and after modification

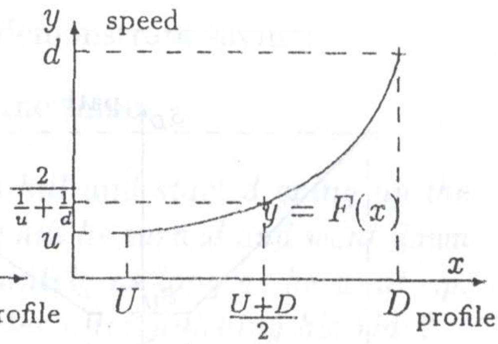
$$y = \frac{u+d}{2} = \frac{10+50}{2} = 30\ (km/h).$$

Let's use preceding process to solve the second, resp. third problem. Then we show that we can some solving make easier and whole consideration generalize.

Let's go back to the second problem. From given problem we can see that the path lengths of cyclist is not important, because both moves passed on a path of the same length. It's also clear that as much the road is falling as faster it is and so the cyclist covers it in shorter time – we can speak about relation of direct proportionality with negative coefficient of proportionality (on pic.) 3 is demonstrated by linear function t). It's evident that the shorter the time is (the time that the cyclist covers the given path), the bigger is the speed of movement, it's the relation of inverse proportionality (on pic. 3 is demonstrated by linear fractional function v).



pic. 3



pic. 4

Let's define $F := v \circ t$. Graph of this function is evidently inverse proportionality with negative coefficient moved to 1. quadrant – see pic. 4. That's so

$$y = F(x) = \frac{k}{x-q}.$$

Because $F(x)$ should go through points (U, u) and (D, d), it's in form

$$y = \frac{\frac{ud(U-D)}{d-u}}{x - \frac{dD-uU}{d-u}}. \tag{2}$$

Now let's use again that the mean speed we are searching for is in fact a speed, which the cyclist would move on fictive plain road $\frac{U+D}{2}$ and let's substitute for x expression $\frac{U+D}{2}$ into relation (2). And after easy modification we get

$$y = \frac{2}{\frac{1}{u}+\frac{1}{d}} = \frac{2}{\frac{1}{10}+\frac{1}{50}} = 16,\overline{6}(km/h).$$

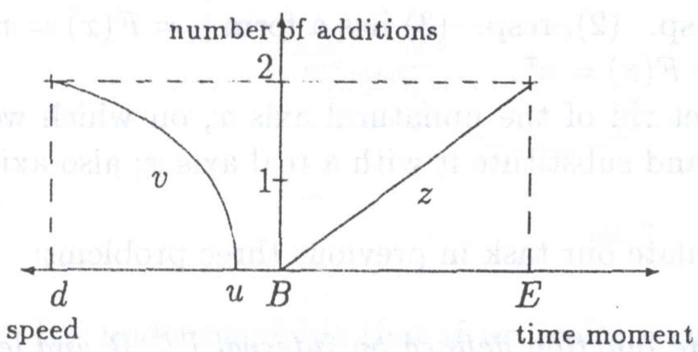
Let's use the same process also for solving the 3. problem. The cyclist changed the speed two times; at the beginning B had speed u, at the end

E speed d. Let's consider mapping z, which adds to „interval" $< B, E >$ on axis of „time moments" (it's an analogy of our interval of roads from 1., resp. 2. problem) an interval on axis number speed additions – evidently it's linear function (it's direct proportionality). Now let's consider mapping v, which adds to the number of additions a speed. Function v has following quality

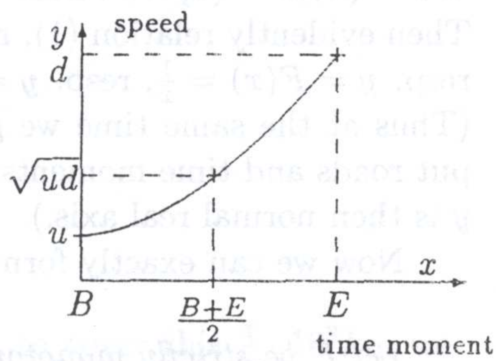
$$\frac{v(0)}{v(h)} = \frac{v(p)}{v(p+h)},$$

for every p, h.

The only continuous function that fits previous relation is an exponential function. The whole situation is demonstrated on pic. 5.



pic. 5



pic. 6

Let's define $F := v \circ z$ and draw a graph of this (again exponential) function – see pic. 6.

That's so

$$y = ka^x.$$

By substitution of points (B, u) a (E, d) into previous equality we get

$$y = u\left(\frac{d}{u}\right)^{\frac{x-B}{E-B}}. \tag{3}$$

Now let's apply assertion that the mean speed is in fact a speed, which the cyclist would have in point $\frac{B+E}{2}$. After modification we get

$$y = \sqrt{ud} = \sqrt{10.50} \doteq 22,36(km/h).$$

Let's repeat what we have done till this time: We had, in each of the three problems given always two values and from them we should have calculated some averages and we know that we have always got different values – probably because the character of each problem was different. Numbers we were searching for we got so that we

(a) found dependance of quantity, which mean values we are searching for (it's speed), on interval which end-points are some two situations ($< U, D >$, resp. $< B, E >$)

(b) we calculated value of this function in „neutral, mean" situation ($\frac{U+D}{2}$, resp. $\frac{B+E}{2}$).

If we consider that exists bijective function of any two intervals (i.e. function $f(x) = \frac{d-c}{b-a}(x-a) + c$ is bijective function of interval $< a, b >$ on interval $< c, d >$), and that a center of the interval $< a, b >$ in this function maps on a center of the interval $< c, d >$. So we can map the interval $< U, D >$ from problem 1, 2, resp. $< B, E >$ from 3. problem on interval $< F^{-1}(n), F^{-1}(d) >$, where F^{-1} is an inverse function to the function F. Then evidently relation (1), resp. (2), resp. (3) has a form $y = F(x) = x$, resp. $y = F(x) = \frac{1}{x}$, resp. $y = F(x) = e^x$.
(Thus at the same time we get rid of the unnatural axis x, on which we put roads and time moments and substitute it with a real axis x; also axis y is then normal real axis.)

Now we can exactly formulate our task in previous three problems:

Let F be strictly monotonic function defined on interval $I \subset R$ and let $a_1, a_2 \in F(I)$ are two points. Find number

$$F\left(\frac{F^{-1}(a_1) + F^{-1}(a_2)}{2}\right). \tag{4}$$

We call this number *quasiarithmetic mean* of numbers a_1, a_2 regarding the function F and that for $F(x) = x$, resp. $F(x) = \frac{1}{x}$, resp. $F(x) = e^x$ has this quasiarithmetic mean special name, and that is mean arithmetic, resp. harmonic, resp. geometric.

Correct formulation of the last sentence in given problems 1., 2. and 3. should be: What is the quasiarithmetic mean of speeds u, d? (Of course we won't tell regarding to what function (then it would be just a mere substitution into formula (5)); the whole problem consists in its finding!)

Now let's demonstrate on one problem that the quasiarithmetic mean has certain extremal character.

Problem 4 *Back wheel tire of a mountainbike is able to ride 10^3 km, the front tire 2.10^3 km. How many km is maximum to ride with these tires?*

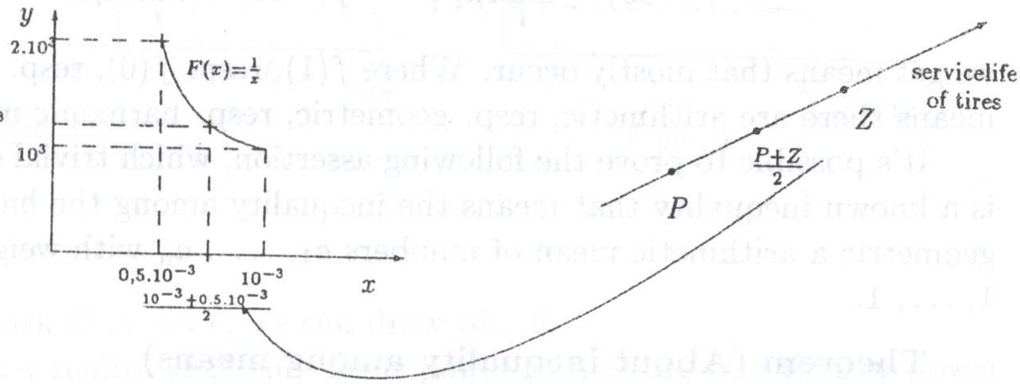
Solving:

It's evident that the more is the tire in back, the less *km* it's able to ride; that is relation of inverse proportionality. Its basic form is $F(x) = \frac{1}{x}$.

Quasiarithmetic mean of numbers 10^3 and 2.10^3 regarding this function (otherwise called harmonic mean) is

$$F\left(\frac{F^{-1}\left(10^3\right)+F^{-1}\left(2.10^3\right)}{2}\right)=\frac{2}{\frac{1}{10^3}+\frac{1}{2.10^3}}=\frac{4}{3}\cdot 10^3\ (km).$$

Whole situation is demonstrated on pic. 7.



pic. 7

It's understandable that if we really want to cover this $\frac{4}{3}\cdot 10^3 km$ we must change the tires after a half of the distance.

Now let's generalize the quasiarithmetic mean in a following definition

Definition (Weighted quasiarithmetic mean)

Let F be strictly monotonic function defined on interval $I \subset R$, let $w_1, \ldots, w_n$ be not negative real numbers $\left(\sum_{i=1}^{n} w_i > 0\right)$ and let $a_1, \ldots, a_n \in F(I)$ be arbitrary numbers. Then the idiom

$$F\left(\frac{\sum_{i=1}^{n} w_i F^{-1}(a_i)}{\sum_{i=1}^{n} w_i}\right) \tag{5}$$

is called the quasiarithmetic weighted mean of numbers $a_1, \ldots, a_n$ regarding the function F with weights $w_1, \ldots, w_n$.

In case that the numbers $a_1, \ldots, a_n$ are all positive and $r \in R$, $r \neq 0$ we get with the option $F(x) = x^{\frac{1}{r}}$ this special case of mean

$$f(r) = \left(\frac{\sum_{i=1}^{n} w_i a_i^r}{\sum_{i=1}^{n} w_i}\right)^{\frac{1}{r}}.$$

If we will also define means

$$f(0) = \exp\left(\frac{\sum\limits_{i=1}^{n} w_i \ln a_i}{\sum\limits_{i=1}^{n} w_i}\right) = \prod_{j=1}^{n} a_j \frac{w_j}{\sum\limits_{i=1}^{n} w_i}$$

and

$$f(+\infty) = \max_i a_i\ , \qquad f(-\infty) = \min_i a_i,$$

we get means that mostly occur. Where $f(1)$, resp. $f(0)$, resp. $f(-1)$, that means there are arithmetic, resp. geometric, resp. harmonic means.

It's possible to prove the following assertion, which trivial consequence is a known inequality that means the inequality among the harmonic, geometric a arithmetic mean of numbers $a_1, \ldots, a_n$ with weights $1, \ldots, 1$.

Theorem (About inequality among means)

Function f(r) is an strictly increasing function of variable r on interval $< -\infty, \infty >$.

This talk gives instructions, how to formulate, solve an give reasons for many practical problems about means, if we take any assertion characterized by functional relation, i.e.

Two same charges Q, are repelled by a power F, which is given in relation

$$F = \frac{kQ^2}{r^2},$$

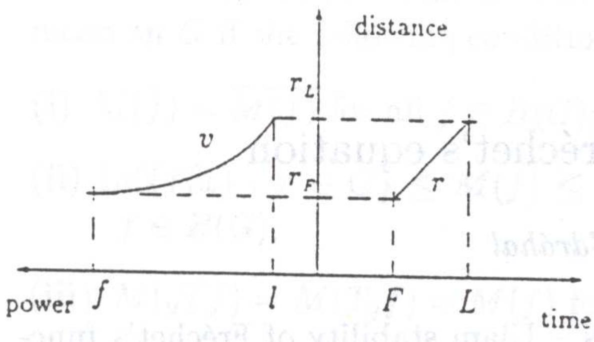
where k is a certain constant and r is a distance between charges.

And now we can make the problem :

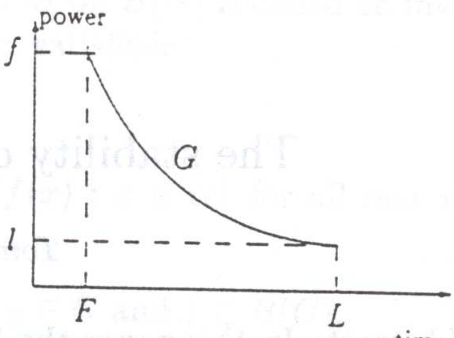
Problem 5 *Let's get over the repelled power of two same charges Q first with a power* $f = 4$ *and later with a power* $l = 1$. *What mean power we worked?*

Solving:

Let's step in a similar way like in previous problems. Let's draw a picture of a superposition of two functions. Let's put time on a right half of the horizontal axis, let's mark the first time moment F and the moment later L. On perpendicular axis we map the distance between charges and finally on the left half of the horizontal axis we map the power. It's evident that the graph will look as following and we now r is direct proportionality and v is quadratic fractional function.



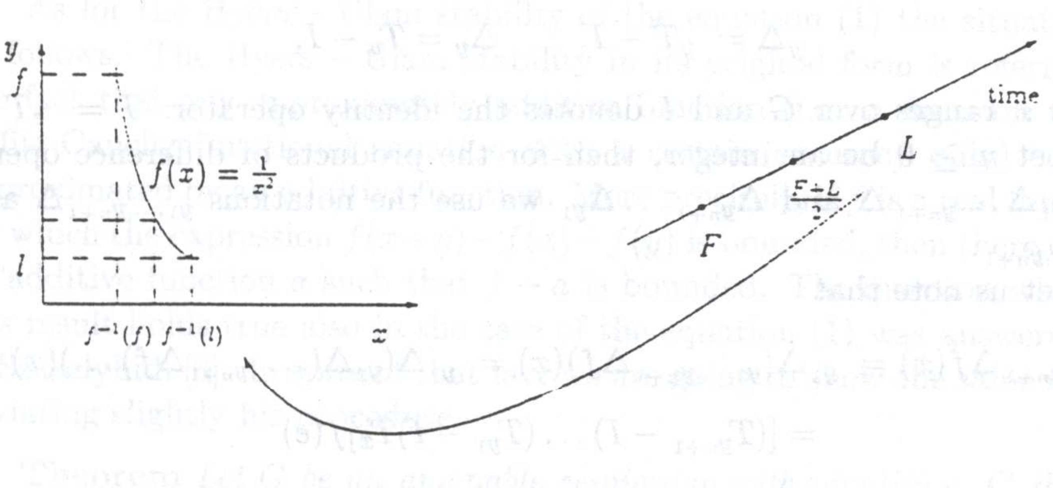
pic. 8



pic. 9

If we mark $G := v \circ r$, we can draw pic. 9.

Now let's realise that the mean power is nothing else than a power in a mean time $\frac{F+L}{2}$. Let's make a transformation of interval $< F, L >$ into interval $< f^{-1}(f), f^{-1}(l) >$ on the horizontal axis x, where $f(x)$ is a basic form of fractional quadratic function G, it's $f(x) = x^{-2}$ (and so $f^{-1}(x) = x^{-\frac{1}{2}}$) - see pic. 10.



pic. 10.

The mean we are searching for is

$$f\left(\frac{f^{-1}(n) + f^{-1}(p)}{2}\right) = \left(\frac{n^{-\frac{1}{2}} + p^{-\frac{1}{2}}}{2}\right)^{-2} = \frac{16}{9}.$$