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The stability of Fréchet's equation

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Abstract: In this paper the Hyers – Ulam stability of Fréchet's functional equation is dealt with. Our approach is motivated by results of L. Székelyhidi (see [2] and [3]) who pointed out that the classical Hyers's theorem on stability of this functional equation holds true (under an auxiliary hypothesis) for functions defined on amenable semigroups.

We use the following notations and terminology. Let G be a semigroup with identity e and let H be a linear space over the rationals. We shall use the operators of left and right translations ${}_yT$ and T_y defined for any function $f : G \to H$ by

$$ {}_yTf(x) = f(yx), \qquad T_yf(x) = f(xy) $$

further, the left and right difference operators ${}_y\Delta$ and Δ_y defined by

$$ {}_y\Delta = {}_yT - I, \qquad \Delta_y = T_y - I, $$

where x ranges over G and I denotes the identity operator: $I = {}_eT = T_e$. Let $n \geq 0$ be an integer, then for the products of difference operators ${}_{y_1}\Delta \ldots {}_{y_{n+1}}\Delta$ and $\Delta_{y_{n+1}} \ldots \Delta_{y_1}$ we use the notations ${}_{y_1,\ldots,y_{n+1}}\Delta$ and $\Delta_{y_1,\ldots,y_{n+1}}$.

Let us note that

$$ {}_{y_1,\ldots,y_{n+1}}\Delta f(x) = {}_{y_1}\Delta({}_{y_2,\ldots,y_{n+1}}\Delta f)(x) = {}_{y_1}\Delta({}_{y_2}\Delta(\ldots {}_{y_{n+1}}\Delta f)\ldots)(x) = $$
$$ = [(T_{y_{n+1}} - I)\ldots(T_{y_1} - I)T_x]f(e) $$

and

$$ \Delta_{y_1,\ldots,y_{n+1}}f(x) = \Delta_{y_{n+1}}(\Delta_{y_1,\ldots,y_n}f)(x) = \Delta_{y_{n+1}}(\Delta_{y_n}(\ldots\Delta_{y_1}f)\ldots)(x) = $$
$$ = [T_x(T_{y_{n+1}} - I)\ldots(T_{y_1} - I)]f(e). $$

Obviously

$$ (T_{y_{n+1}} - I)\ldots(T_{y_1} - I) = \sum_{\alpha_{n+1},\ldots,\alpha_1=0}^{1} (-1)^{\alpha_{n+1}+\ldots+\alpha_1}\, T_{y_{n+1}^{(1-\alpha_{n+1})}\ldots y_1^{(1-\alpha_1)}} $$

Now we recall the notation of invariant mean and amenability.

Let $B(G)$ denotes the space of all bounded complex-valued functions on G with sup–norm. A linear functional M on $B(G)$ is called an invariant mean on G if the following conditions are satisfied:

(i) $M(\overline{f}) = \overline{M(f)}$ for all $f \in B(G)$;

(ii) $\inf\{f(x) : x \in G\} \leq M(f) \leq \sup\{f(x) : x \in G\}$ for all real–valued $f \in B(G)$;

(iii) $M({}_yTf) = M(T_yf) = M(f)$ for all $y \in G$ and $f \in B(G)$.

We note the following properties of a mean which are easily verified: $M(1) = 1$, $M(c) = c$ for a constant $c \in \mathbf{C}$ and the norm of the functional M is one.

If an invariant mean exists, we call G amenable. (An extensive survey of criterions of amenability together with an ample bibliography of the subject may be found in [1]. In particular it is known that every Abelian semigroup is amenable, but amenability is much weaker condition than commutativity.)

Fréchet's functional equation is the following one

$$\Delta_{y_1,\dots,y_{n+1}} f(x) = 0 \tag{1}$$

for all $x, y_1, \dots, y_{n+1} \in G$, where $f : G \to H$ is an unknown function.

As for the Hyers - Ulam stability of the equation (1) the situation is as follows. The Hyers – Ulam stability in its original form is referred to the fact that any approximately additive function (i.e. a function which fulfils Cauchy functional equation with a certain accuracy only) can be approximated by an additive function. More precisely, if f is a real function for which the expression $f(x+y) - f(x) - f(y)$ is bounded, then there exists an additive function a such that $f - a$ is bounded. The question whether this result holds true also in the case of the equation (1) was answered by L. Székelyhidi [3]. In spite of that fact we are going to show the same result deviating slightly his procedure.

Theorem *Let G be an amenable semigroup with identity e, $\mathbf{C}$ the set of all complex numbers and $f : G \to \mathbf{C}$ a function for which the function $(x, y_1, \dots, y_{n+1}) \to \Delta_{y_1,\dots,y_{n+1}} f(x)$ is bounded. Then there exists a function $F : G \to \mathbf{C}$ satisfying Fréchet's functional equation (1) uniformly approximating the function f.*

Proof. Let M stand for an invariant mean on G. Then for all $f \in B(G)$ the function $m(f)$ defined by

$$m(f)(x) = (-1)^n M_{y_1} \dots M_{y_n} (\Delta_{y_1,\dots,y_n,x} f(e)),$$

where the subscript y_i next to M, $i = 1, \dots, n$, indicates the fact that M is applied to a function of the variable y_i and $M_{y_1} \dots M_{y_n}$ denotes the product of invariant means, is obviously bounded.

We can suppose without loss of the generality that $f(e) \neq 0$. In what follows we are going to check that the function

$$F(x) := f(x) - f(e) - m(f)(x)$$

is a solution of Fréchet's equation (1). Then the fact that $m(f)$ is bounded ends the proof of this theorem.

Let us start with some lemmas.

Lemma 1 *Let $f : G \to \mathbf{C}$ be arbitrary. Then for all $x, y_1, \dots, y_{n+1} \in G$ holds true*

$$ {}_{y_1,\dots,y_{n+1}}\Delta f(x) = \Delta_{x,y_1,\dots,y_n} f(y_{n+1}) - \Delta_{x,y_1,\dots,y_n} f(e) + \Delta_{y_1,\dots,y_{n+1}} f(e), \tag{2}$$

$$\Delta_{y_1,\dots,y_{n+1}} f(x) = {}_{y_2,\dots,y_{n+1},x}\Delta f(y_1) - {}_{y_2,\dots,y_{n+1},x}\Delta f(e) + {}_{y_1,\dots,y_{n+1}}\Delta f(e). \tag{3}$$

Proof. It is obvious that

$$(T_{y_{n+1}} - I)(T_{y_n} - I) \dots (T_{y_1} - I)(T_x - I) = T_{y_{n+1}}(T_{y_n} - I) \dots (T_{y_1} - I)(T_x - I) -$$
$$- (T_{y_n} - I) \dots (T_{y_1} - I)(T_x - I)$$

analogously

$$(T_{y_{n+1}} - I)(T_{y_n} - I) \dots (T_{y_1} - I)(T_x - I) = (T_{y_{n+1}} - I)(T_{y_n} - I) \dots (T_{y_1} - I)T_x -$$
$$- (T_{y_{n+1}} - I)(T_{y_n} - I) \dots (T_{y_1} - I).$$

In virtue of the facts presented at the beginning of this paper we have just shown that the following equality holds true

$$\Delta_{x,y_1,\dots,y_n} f(y_{n+1}) - \Delta_{x,y_1,\dots,y_n} f(e) = {}_{y_1,\dots,y_{n+1}}\Delta f(x) - {}_{y_1,\dots,y_{n+1}}\Delta f(e).$$

Since

$${}_{y_1,\dots,y_{n+1}}\Delta f(e) = \Delta_{y_1,\dots,y_{n+1}} f(e),$$

we get

$${}_{y_1,\dots,y_{n+1}}\Delta f(x) = \Delta_{x,y_1,\dots,y_n} f(y_{n+1}) - \Delta_{x,y_1,\dots,y_n} f(e) + \Delta_{y_1,\dots,y_{n+1}} f(e).$$

Putting $y_{n+1} := x,\ x := y_1,\ y_i := y_{i+1}$ for $i = 1, \dots, n$ in the last equation we get immediately (3) and the lemma is proved.

Lemma 2 *Let the assumption of the above theorem are fulfilled. Then*

$$M_{y_1} \dots M_{y_k} \left[{}_{t_1,\dots,t_{n+1}}\Delta f \left(y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)} \right) \right] = 0, \tag{4}$$

where $\alpha_i = 0$ or 1, $i = 1, \dots, k$, $\alpha_k \dots \alpha_1 = 0$ for all $y_1, \dots, y_k, t_1, \dots, t_{n+1} \in G$.

Proof. From our assumption on f and from (2) it follows that also the function $\left(y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}, t_1, \dots, t_{n+1} \right) \to {}_{t_1,\dots,t_{n+1}}\Delta f \left(y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)} \right)$ is bounded, i.e. (4) is well defined. Further, we have for all $y_1, \dots, y_k, t_1, \dots, t_{n+1}$:

$$M_{y_1} \dots M_{y_k} \left[{}_{t_1,\dots,t_{n+1}}\Delta f \left(y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)} \right) \right] = \\ = M_{y_1} \dots M_{y_k} \{ [(T_{t_{n+1}} - I) \dots (T_{t_1} - I)$$

$$T_{y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}}] f(e) \} =$$
$$= M_{y_1} \dots M_{y_k} \{ [(T_{t_{n+1}} - I) \dots (T_{t_2} - I)(T_{t_1 y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}} - T_{y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}}] f(e) \}$$
$$= M_{y_1} \dots M_{y_k} \{ [(T_{t_{n+1}} - I) \dots (T_{t_2} - I)(T_{t_1 y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}} - I) -$$
$$- (T_{t_{n+1}} - I) \dots (T_{t_2} - I)(T_{y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}} - I)] f(e) \} =$$
$$= M_{y_1} \dots M_{y_k} \{ [(T_{t_{n+1}} - I) \dots (T_{t_2} - I)$$

$$(T_{t_1 y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}} - I)] f(e) \} - M_{y_1} \dots M_{y_k} \{ [(T_{t_{n+1}} - I) \dots (T_{t_2} - I)$$

$$(T_{y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}} - I)] f(e) \} = 0,$$

because $M_{y_1}, \dots, M_{y_k}$ are invariants (and $M_{y_1} \dots M_{y_k}$ is the product of these ones).

Lemma 3 *Let again the assumptions of the theorem are fulfilled. Then for all $x, y_1, \dots, y_k, u_1, \dots, u_{n+1} \in G$ the following identity is valid:*

$$M_{y_1} \dots M_{y_k} [{}_{u_1,\dots,u_{n+1}}\Delta (\Delta_{y_1,\dots,y_k} f)(x)] = (-1)^k \, {}_{u_1,\dots,u_{n+1}}\Delta f(x).$$

Proof. This is verified by induction on n. For $n = 0$ we have

$$M_{y_1} \dots M_{y_k}[{}_{u_1}\Delta(\Delta_{y_1,\dots,y_k} f)(x)] = M_{y_1} \dots M_{y_k}[\Delta_{y_1,\dots,y_k} f(u_1 x) - \Delta_{y_1,\dots,y_k} f(x)] =$$

$$= M_{y_1} \dots M_{y_k}\{[T_{u_1 x}(T_{y_k} - I) \dots (T_{y_1} - I) - T_x(T_{y_k} - I) \dots (T_{y_1} - I)]f(e)\} =$$

$$= M_{y_1} \dots M_{y_k}\{[(T_{u_1 x} - T_x)(T_{y_k} - I) \dots (T_{y_1} - I)]f(e)\} =$$

$$= M_{y_1} \dots M_{y_k}\{[(T_{u_1} - I)T_x(T_{y_k} - I) \dots (T_{y_1} - I)]f(e)\} =$$

$$= M_{y_1} \dots M_{y_k}\{[(T_{u_1} - I)(T_x - I)(T_{y_k} - I) \dots (T_{y_1} - I) +$$

$$+ (T_{u_1} - I)(T_{y_k} - I) \dots (T_{y_1} - I)]f(e)\} =$$

$$= M_{y_1} \dots M_{y_k}\{[(T_{u_1} - I)(T_x - I) \sum_{\alpha_k,\dots,\alpha_1=0}^{1} (-1)^{\alpha_k+\dots+\alpha_1} T_{y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}} +$$

$$+ (T_{u_1} - I) \sum_{\alpha_k,\dots,\alpha_1=0}^{1} (-1)^{\alpha_k+\dots+\alpha_1} T_{y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}}]f(e)\} =$$

$$= M_{y_1} \dots M_{y_k}\{[\sum_{\substack{\alpha_k,\dots,\alpha_1=0 \\ \alpha_k \dots \alpha_1=0}}^{1} (-1)^{\alpha_k+\dots+\alpha_1} (T_{u_1} - I)(T_x - I)T_{y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}} +$$

$$+ \sum_{\substack{\alpha_k,\dots,\alpha_1=0 \\ \alpha_k \dots \alpha_1=0}}^{1} (-1)^{\alpha_k+\dots+\alpha_1} (T_{u_1} - I)T_{y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}} +$$

$$+ (-1)^k(T_{u_1} - I)(T_x - I)I + (-1)^k(T_{u_1} - I)]f(e)\} =$$

$$= M_{y_1} \dots M_{y_k}\{ \sum_{\substack{\alpha_k,\dots,\alpha_1=0 \\ \alpha_k \dots \alpha_1=0}}^{1} (-1)^{\alpha_k+\dots+\alpha_1} {}_{x,u_1}\Delta f\left(y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}\right) +$$

$$+ \sum_{\substack{\alpha_k,\dots,\alpha_1=0 \\ \alpha_k \dots \alpha_1=0}}^{1} (-1)^{\alpha_k+\dots+\alpha_1} {}_{u_1}\Delta f\left(y_k^{(1-\alpha_k)} \dots y_1^{(1-\alpha_1)}\right) + (-1)^k {}_{u_1}\Delta f(x)\} = 0,$$

because of lemma 2 and the linearity of the mean.

Now suppose that the statement holds true for n; then we get

$$M_{y_1} \dots M_{y_k}[{}_{u_1,\dots,u_{n+1}}\Delta(\Delta_{y_1,\dots,y_k} f)(x)] =$$

$$= M_{y_1} \dots M_{y_k}[{}_{u_1}\Delta({}_{u_2,\dots,u_{n+1}}\Delta(\Delta_{y_1,\dots,y_k} f))(x)] =$$

$$= M_{y_1} \dots M_{y_k}[{}_{u_2,\dots,u_{n+1}}\Delta(\Delta_{y_1,\dots,y_k} f)(u_1 x) - {}_{u_2,\dots,u_{n+1}}\Delta(\Delta_{y_1,\dots,y_k} f)(x)] =$$

$$= (-1)^k {}_{u_2,\dots,u_{n+1}}\Delta f(u_1 x) - (-1)^k {}_{u_2,\dots,u_{n+1}}\Delta f(x) = (-1)^k {}_{u_1,\dots,u_{n+1}}\Delta f(x)$$

and the proof ends.

Now we can go back to the proof of our theorem. We have for all $x, u_1, \ldots, u_{n+1} \in G$

$$
\begin{aligned}
{}_{u_1,\ldots,u_{n+1}}\Delta F(x) &= {}_{u_1,\ldots,u_{n+1}}\Delta f(x) - {}_{u_1,\ldots,u_{n+1}}\Delta f(e) + \\
&+ (-1)^{n+1}{}_{u_1,\ldots,u_{n+1}}\Delta M_{y_1}\ldots M_{y_n}[\Delta_{y_1,\ldots,y_n} f(x) - \Delta_{y_1,\ldots,y_n} f(e)] = \\
&= {}_{u_1,\ldots,u_{n+1}}\Delta f(x) - {}_{u_1,\ldots,u_{n+1}}\Delta f(e) + \\
&+ (-1)^{n+1} M_{y_1}\ldots M_{y_n}[{}_{u_1,\ldots,u_{n+1}}\Delta(\Delta_{y_1,\ldots,y_n} f)(x)] + \\
&+ (-1)^n M_{y_1}\ldots M_{y_n}[{}_{u_1,\ldots,u_{n+1}}\Delta(\Delta_{y_1,\ldots,y_n} f)(e)] = \\
&= {}_{u_1,\ldots,u_{n+1}}\Delta f(x) - {}_{u_1,\ldots,u_{n+1}}\Delta f(e) + \\
&+ (-1)^{n+1}(-1)^n\, {}_{u_1,\ldots,u_{n+1}}\Delta f(x) + (-1)^n(-1)^n\, {}_{u_1,\ldots,u_{n+1}}\Delta f(e) = 0,
\end{aligned}
$$

because of lemma 3. In virtue of (3) we finally get that also

$$\Delta_{u_1,\ldots,u_{n+1}} F(x) = 0 \text{ for all } x, u_1, \ldots, u_{n+1} \in G,$$

i.e. $F(x)$ is a solution of Fréchet's equation (1) and our theorem is proved.

References

[1] Greenleaf, P.: Invariant means on topological groups, Van Nostrand, 1969

[2] Székelyhidi, L.: Note on Hyers's theorem, C. R. Math. Rep. Sci. Canada 8 (1986), 127 - 129

[3] Székelyhidi, L.: Fréchet's equation and Hyers theorem on noncommutative semi-groups, Annal. Pol. Math. XLVIII (1988), 183 - 189